Bifurcations of Wavefronts on an r-corner

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Abstract

We introduce the notion of *reticular Legendrian unfoldings* in order to investigate stabilities of bifurcations of wavefronts generated by a hypersurface germ with a boundary, a corner, or an r-corner in a smooth n dimensional manifold. We define several stabilities of reticular Legendrian unfoldings and prove that they and the stabilities of corresponding generating families are all equivalent and give a classification of generic bifurcations of wavefronts in the cases $r=0, n \le 5$ and $r=1, n \le 3$ respectively.

1 Introduction

In [5] K.Jänich explained the wavefront propagation mechanism on a manifold which is completely described by a positive and positively homogeneous *Hamiltonian function* on the cotangent bundle and investigated the local gradient models given by the ray length function. Caustics and Wavefronts generated by an initial wavefront which is a hypersurface germ without boundary in the manifold is investigated as Lagrangian and Legendrian singularities by V.I.Arnold (cf., [1]).

In this paper and its prequel [10], we investigate stabilities and a genericity of bifurcations of wavefronts generated by a hypersurface germ with *an r-corner*. Wavefronts generated by all edges of the hypersurface at a time give *a contact regular r-cubic configuration* on the 1-jet bundle. All wavefronts around a time give a one-parameter family of contact regular r-cubic configurations on the 1-jet bundle. In order to consider families like these, we shall introduce the notion of *unfolded contact regular r-cubic configurations* on the big 1-jet bundle. A wavefront of an unfolded contact regular r-cubic configuration is the big front of the corresponding one-parameter family of contact regular r-cubic configurations. We shall consider their generating families and equivalence relations.
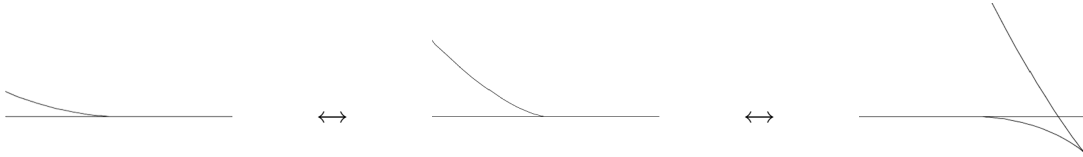

Figure 1: A generic bifurcation of wavefronts on a boundary $^1B_3$

In order to investigate stabilities and a genericity of unfolded contact regular r-cubic configurations, we introduce the notion of *reticular Legendrian unfoldings* which is a generalised notion of Legendrian unfoldings given by S.Izumiya (cf., [2]) for our situation. We

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shall define several stabilities of reticular Legendrian unfoldings and prove that they and stabilities of their generating families are all equivalent. We shall also classify generic reticular Legendrian unfoldings and give all figures of their wavefronts in the case $r = 1, n \leq 3$.

This paper consists of four sections. In Section 2 we shall give a motivation for this paper and give a review of stabilities under the *reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence relation* of function germs which plays an important role as generating families of reticular Legendrian unfoldings. We shall also give a review of the theory of contact regular r-cubic configurations. In Section 3 we shall introduce the notion of reticular Legendrian unfoldings and consider their generating families. In Section 4 we shall consider several stabilities of reticular Legendrian unfoldings. In Section 5 we shall reduce our investigation to finitely dimensional jet spaces and give a classification of generic reticular Legendrian unfoldings.

All maps considered here are differentiable of class C^{∞} unless stated otherwise.

2 Preliminary

2.1 Propagation mechanism of wavefronts

Let us start with the propagation mechanism of wavefronts generated by a hypersurface germ V^0 with an r-corner in an $(n+1)(=r+k+1)$-dimensional smooth manifold M which is given in [5]. Let $H : T^*M \backslash 0 \to \mathbb{R}$ be a fixed Hamiltonian function, which we suppose that $H(\lambda\xi) = \lambda H(\xi)$ for all $\lambda > 0$ and $\xi \in T^*M \backslash 0$. For example, consider a Riemann manifold M and H be the length of a covector in $T^*M \backslash 0$.

The manifold $E = H^{-1}(1)$ has the contact structure defined by the restriction of the canonical 1-form on T^*M and the projection $\pi : E \to M$ is a spherical cotangent bundle.

Let $\mathbb{H}^r = \{(x_1, \cdots, x_r) \in \mathbb{R}^r | x_1 \geq 0, \cdots, x_r \geq 0\}$ be an r-corner, $\xi_0 \in E$, $t_0 \geq 0$, and V^0 be a hypersurface germ which defined by the image of the immersion $\iota : (\mathbb{H}^r \times \mathbb{R}^k, 0) \to M$ such that $\iota(0) = \pi(\xi_0)$ $\xi_0|_{T_{\iota(0)}V^0} = 0$. Let η_0 be the image of the phase flow of the Hamiltonian vector field X_H at (t_0, ξ_0). Since the flow preserves values of H and the contact structure on E, it induces the contact embedding germs $C_t : (E, \xi_0) \to E$ for t around t_0 which depends smoothly on t. We define the σ-edge V_{σ}^0 of V^0 by $V^0 \cap \{x_{\sigma} = 0\}$ for $\sigma \subset I_r = \{1, \ldots, r\}$. Let L_{σ}^0 be the initial covectors in E generated by V_{σ}^0 to conormal directions, that is

$$L_{\sigma}^0 = \{\xi_q \in E \mid q \in V_{\sigma}^0, \ \xi_q|_{T_q V_{\sigma}^0} = 0\}.$$

We may regard L_{σ}^0 with the lift of V_{σ}^0. We also define that $L_{\sigma,t} = C_t(L_{\sigma}^0)$ for $\sigma \subset I_r, t \in (\mathbb{R}, t_0)$. Then the wavefront $W_{\sigma,t}$ generated by V_{σ}^0 to conormal directions at time t is given by $W_{\sigma,t} = \pi(L_{\sigma,t})$ for $\sigma \subset I_r, t \in (\mathbb{R}, t_0)$.

We are concerned with stabilities and a genericity of bifurcations of wavefronts $\{W_{\sigma,t}\}_{\sigma \subset I_r}$ for t around t_0 with respect to perturbations of V^0.

Since we shall discuss local situations, we may identify (E, ξ_0) with $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$, $\pi : (E, \eta_0) \to (M, \pi(\eta_0))$ with $\pi : (J^1(\mathbb{R}^n, \mathbb{R}), 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$, and $t_0 = 0$, where $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ is the natural Legendrian bundle which is introduced in Section 2.3. Then C_t is identified with $C_t : (J^1(\mathbb{R}^n, \mathbb{R}), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$ for $t \in (\mathbb{R}, 0)$ with $C_0(0) = 0$. The coordinate system on (E, ξ_0) may be chosen that L_{σ}^0 is given:

$$L_{\sigma}^0 = \{(q, z, p) \in (J^1(\mathbb{R}^n, \mathbb{R}), 0)|q_{\sigma} = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = z = 0, q_{I_r - \sigma} \geq 0\}$$
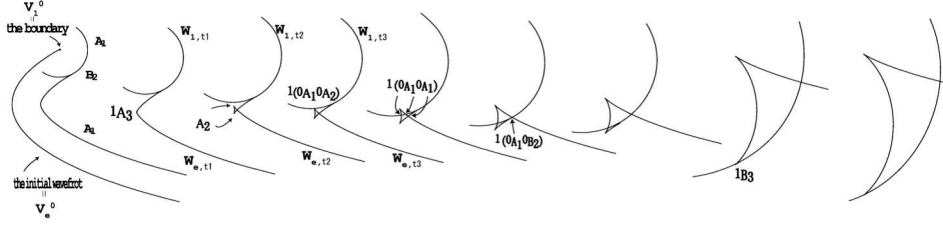
for each $\sigma \subset I_r$.

Figure 2: The initial wavefront V^0 with a boundary and generated wavefronts ($e = \emptyset$, $t_1 < t_2 < t_3$)

We shall consider one-parameter families of contact regular r-cubic configurations $\{L_{\sigma,t}\}_{\sigma \subset I_r, t \in (\mathbb{R},0)}$.

2.2 Stabilities of unfoldings

We review the main results of the theory of function germs with respect to the reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence relation given in [10].

We denote by $\mathcal{E}(r;k_1,r;k_2)$ the set of all germs at 0 in $\mathbb{H}^r \times \mathbb{R}^{k_1}$ of smooth maps $\mathbb{H}^r \times \mathbb{R}^{k_1} \to \mathbb{H}^r \times \mathbb{R}^{k_2}$ and set $\mathfrak{M}(r;k_1,r;k_2) = \{f \in \mathcal{E}(r;k_1,r;k_2) | f(0) = 0\}$. We denote $\mathcal{E}(r;k_1,k_2)$ for $\mathcal{E}(r;k_1,0;k_2)$ and denote $\mathfrak{M}(r;k_1,k_2)$ for $\mathfrak{M}(r;k_1,0;k_2)$.

If $k_2 = 1$ we write simply $\mathcal{E}(r;k)$ for $\mathcal{E}(r;k,1)$ and $\mathfrak{M}(r;k)$ for $\mathfrak{M}(r;k,1)$. Then $\mathcal{E}(r;k)$ is an $\mathbb{R}$-algebra in the usual way and $\mathfrak{M}(r;k)$ is its unique maximal ideal. We also denote by $\mathcal{E}(k)$ for $\mathcal{E}(0;k)$ and $\mathfrak{M}(k)$ for $\mathfrak{M}(0;k)$.

We denote by $J^l(r+k,p)$ the set of l-jets at 0 of germs in $\mathcal{E}(r;k,p)$. There are natural projections:

$$\pi_l : \mathcal{E}(r;k,p) \longrightarrow J^l(r+k,p), \pi_{l_2}^{l_1} : J^{l_1}(r+k,p) \longrightarrow J^{l_2}(r+k,p) \ (l_1 > l_2).$$

We write $j^l f(0)$ for $\pi_l(f)$ for each $f \in \mathcal{E}(r;k,p)$.

Let $(x,y) = (x_1, \cdots, x_r, y_1, \cdots, y_k)$ be a fixed coordinate system of $(\mathbb{H}^r \times \mathbb{R}^k, 0)$. We denote by $\mathcal{B}(r;k)$ the group of diffeomorphism germs $(\mathbb{H}^r \times \mathbb{R}^k, 0) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$ of the form:

$$\phi(x,y) = (x_1 \phi_1^1(x,y), \cdots, x_r \phi_1^r(x,y), \phi_2^1(x,y), \cdots, \phi_2^k(x,y)).$$

We say that $f_0, g_0 \in \mathcal{E}(r;k)$ are *reticular $\mathcal{K}$-equivalent* if there exist $\phi \in \mathcal{B}(r;k)$ and a unit $a \in \mathcal{E}(r;k)$ such that $g_0 = a \cdot f_0 \circ \phi$. We call (ϕ, a) *a reticular $\mathcal{K}$-isomorphism from f_0 to g_0*.

We denote by $\mathcal{B}_n(r;k+n)$ the group of diffeomorphism germs $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0) \to (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$ of the form:

$$\phi(x,y,u) = (x_1 \phi_1^1(x,y,u), \cdots, x_r \phi_1^r(x,y,u), \phi_2^1(x,y,u), \cdots, \phi_2^k(x,y,u), \phi_3^1(u), \ldots, \phi_3^n(u)).$$

We denote $\phi(x,y,u) = (x\phi_1(x,y,u), \phi_2(x,y,u), \phi_3(u))$ and denote other notations analogously.

We say that $f, g \in \mathcal{E}(r;k+n)$ are *reticular $\mathcal{P}$-$\mathcal{K}$-equivalent* if there exist $\Phi \in \mathcal{B}_n(r;k+n)$ and a unit $\alpha \in \mathcal{E}(r;k+n)$ such that $g = \alpha \cdot f \circ \Phi$. We call (Φ, α) *a reticular $\mathcal{P}$-$\mathcal{K}$-isomorphism*

from f to g.

We say that a function germ $f_0 \in \mathfrak{M}(r;k)$ is *reticular $\mathcal{K}$-l-determined* if all function germ which has same l-jet of f_0 is reticular $\mathcal{K}$-equivalent to f_0.

Lemma 2.1 *Let $f_0(x,y) \in \mathfrak{M}(r;k)$ and let*

$$\mathfrak{M}(r;k)^{l+1} \subset \mathfrak{M}(r;k)(\langle f_0, x_1 \frac{\partial f_0}{\partial x_1}, \cdots, x_r \frac{\partial f_0}{\partial x_r} \rangle + \mathfrak{M}(r;k) \langle \frac{\partial f_0}{\partial y_1}, \cdots, \frac{\partial f_0}{\partial y_k} \rangle) + \mathfrak{M}(r;k)^{l+2},$$

then f_0 is reticular $\mathcal{K}$-l-determined. Conversely if $f_0(x,y) \in \mathfrak{M}(r;k)$ is reticular $\mathcal{K}$-l-determined, then

$$\mathfrak{M}(r;k)^{l+1} \subset \langle f_0, x_1 \frac{\partial f_0}{\partial x_1}, \cdots, x_r \frac{\partial f_0}{\partial x_r} \rangle_{\mathcal{E}(r;k)} + \mathfrak{M}(r;k) \langle \frac{\partial f_0}{\partial y_1}, \cdots, \frac{\partial f_0}{\partial y_k} \rangle.$$

We denote $\frac{\partial f_0}{\partial y} = (\frac{\partial f_0}{\partial y_1}, \cdots, \frac{\partial f_0}{\partial y_k})$ and denote other notations analogously.

In convenience, we denote an unfolding of a function germ $f(x,y,u) \in \mathfrak{M}(r;k+n)$ by $F(x,y,t,u) \in \mathfrak{M}(r;k+m+n)$.

Let $F(x,y,t,u) \in \mathfrak{M}(r;k+m_1+n)$ and $G(x,y,s,u) \in \mathfrak{M}(r;k+m_2+n)$ be unfoldings of $f(x,y,u) \in \mathfrak{M}(r;k+n)$.

A reticular t-$\mathcal{P}$-$\mathcal{K}$-f-morphism from F to G is a pair (Φ, α), where $\Phi \in \mathfrak{M}(r;k+m_2+n, r;k+m_1+n)$ and α is an unit of $\mathcal{E}(r;k+m_2+n)$, satisfying the following conditions:
(1) Φ can be written in the form:

$$\Phi(x,y,s,u) = (x\phi_1(x,y,s,u), \phi_2(x,y,s,u), \phi_3(s), \phi_4(s,u)),$$

(2) $\Phi|_{\mathbb{H}^r \times \mathbb{R}^{k+n}} = id_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$, $\alpha|_{\mathbb{H}^r \times \mathbb{R}^{k+n}} \equiv 1$
(3) $G(x,y,s,u) = \alpha(x,y,s,u) \cdot F \circ \Phi(x,y,s,u)$ for all $(x,y,s,u) \in (\mathbb{H}^r \times \mathbb{R}^{k+m_2+n}, 0)$.
If there exists a reticular t-$\mathcal{P}$-$\mathcal{K}$-f-morphism from F to G, we say that *G is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-induced from F*. If $m_1 = m_2$ and Φ is invertible, we call (Φ, α) *a reticular t-$\mathcal{P}$-$\mathcal{K}$-f-isomorphism from F to G* and we say that *F is reticular t-$\mathcal{P}$-$\mathcal{K}$-f*-equivalent to G.

We say that $F(x,y,t,u), G(x,y,t,u) \in \mathcal{E}(r;k+m+n)$ are *reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent* if there exist Φ of $\mathcal{B}(r;k+m+n)$ and a unit $\alpha \in \mathcal{E}(r;k+m+n)$ such that
(1) Φ can be written in the form:

$$\Phi(x,y,t,u) = (x_1\phi_1(x,y,t,u), \phi_2(x,y,t,u), \phi_3(t), \phi_4(t,u)), \tag{2.1}$$

(2) $G(x,y,t,u) = \alpha(x,y,t,u) \cdot F \circ \Phi(x,y,t,u)$ for all $(x,y,t,u) \in (\mathbb{H}^r \times \mathbb{R}^{k+m+n}, 0)$.
We call (Φ, α) *a reticular t-$\mathcal{P}$-$\mathcal{K}$-isomorphism from F to G*.

Let $f(x,y,u) \in \mathcal{E}(r;k+n)$ and set $z = j^l f(0)$. Let $O^l_{r\mathcal{P}\text{-}\mathcal{K}}(z)$ be the submanifold of $J^l(r+k+n,1)$ consists of the image by π_l of the orbit of the reticular $\mathcal{P}$-$\mathcal{K}$-equivalence of f. Then we have that

$$T_z(O^l_{r\mathcal{P}\text{-}\mathcal{K}}(z)) = \pi_l(\langle f, x\frac{\partial f}{\partial x} \rangle_{\mathcal{E}(r;k+n)} + \mathfrak{M}(r;k+n) \langle \frac{\partial f}{\partial y} \rangle + \mathfrak{M}(n) \langle \frac{\partial f}{\partial u} \rangle). \tag{2.2}$$

4

Let U be a neighbourhood of 0 in $\mathbb{R}^{r+k+m+n}$ and let $F:U \to \mathbb{R}$ be a smooth function and l be a non-negative integer. We define the smooth map germ

$$j_1^l F : U \longrightarrow J^l(r+k+n,1)$$

as the follow: For $(x,y,t,u) \in U$ we set $j_1^l F(x,y,t,u)$ by the l-jet of the function germ $F_{(x,y,t,u)} \in \mathfrak{M}(r;k+n)$ at 0, where $F_{(x,y,t,u)}$ is given by $F_{(x,y,t,u)}(x',y',u') = F(x+x',y+y',t,u+u') - F(x,y,t,u)$.

Definition 2.2 We define stabilities of unfoldings. Let $F(x,y,t,u) \in \mathfrak{M}(r;k+m+n)$ be an unfolding of $f(x,y,u) \in \mathfrak{M}(r;k+n)$.

Let l be a non-negative integer and $z = j^l f(0)$. We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-l-transversal unfolding of f* if the $j_1^l F|_{x=0}$ at 0 is transversal to $O_{r\mathcal{P}\text{-}\mathcal{K}}^l(z)$.

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of f* if the following condition holds: For any neighbourhood U of 0 in $\mathbb{R}^{r+k+m+n}$ and any representative $\tilde{F} \in C^\infty(U,\mathbb{R})$ of F, there exists a neighbourhood $N_{\tilde{F}}$ of $\tilde{F}$ in $C^\infty(U,\mathbb{R})$ with the C^∞-topology such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+m+n}}$ at $(0,y_0,t_0,u_0)$ is reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent to F for some $(0,y_0,t_0,u_0) \in U$.

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-versal unfolding of f* if any unfolding of f is reticular t-$\mathcal{P}$-$\mathcal{K}$-f-induced from F.

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally versal* if

$$\mathcal{E}(r;k+n) = \langle f, x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathcal{E}(r;k+n)} + \langle \frac{\partial f}{\partial u} \rangle_{\mathcal{E}(n)} + \langle \frac{\partial F}{\partial t}|_{t=0} \rangle_{\mathbb{R}}.$$

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable* if

$$\mathcal{E}(r;k+m+n) = \langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle_{\mathcal{E}(r;k+m+n)} + \langle \frac{\partial F}{\partial u} \rangle_{\mathcal{E}(m+n)} + \langle \frac{\partial F}{\partial t} \rangle_{\mathcal{E}(m)}. \tag{2.3}$$

We say that F is *reticular t-$\mathcal{P}$-$\mathcal{K}$-homotopically stable* if for any smooth path-germ $(\mathbb{R},0) \to \mathcal{E}(r;k+m+n), \tau \mapsto F_\tau$ with $F_0 = F$, there exists a smooth path-germ $(\mathbb{R},0) \to \mathcal{B}(r;k+m+n) \times \mathcal{E}(r;k+m+n), \tau \mapsto (\Phi_\tau, \alpha_\tau)$ with $(\Phi_0, \alpha_0) = (id,1)$ such that each (Φ_τ, α_τ) is a reticular t-$\mathcal{P}$-$\mathcal{K}$-isomorphism and $F_\tau = \alpha_\tau \cdot F_0 \circ \Phi_\tau$ for $\tau \in (\mathbb{R},0)$.

Theorem 2.3 *Let f be an unfolding of $f_0(x,y) \in \mathfrak{M}(r;k)$ and $F(x,y,t,u) \in \mathfrak{M}(r;k+m+n)$ be an unfolding of $f(x,y,u) \in \mathfrak{M}(r;k+n)$. Then following are equivalent.*
(1) There exists a non-negative number l such that f_0 is reticular $\mathcal{K}$-l-determined and F is reticular t-$\mathcal{P}$-$\mathcal{K}$-l'-transversal for $l' \geq lm + l + m + 1$.
(2) F is reticular t-$\mathcal{P}$-$\mathcal{K}$-stable.
(3) F is reticular t-$\mathcal{P}$-$\mathcal{K}$-versal.
(4) F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally versal.
(5) F is reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimally stable.
(6) F is reticular t-$\mathcal{P}$-$\mathcal{K}$-homotopically stable.

The classification list of reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfoldings in $\mathfrak{M}(r;k+1+n)$ with $r=0, n \leq 5$ or $r=1, n \leq 3$ are given in [10, p.201].

Theorem 2.4 (Cf., [10, Theorem 4.7]) *Let $r=0, n \leq 5$ or $r=1, n \leq 3$ and U be a neighbourhood of 0 in $\mathbb{H}^r \times \mathbb{R}^{k+1+n}$. Then there exists a residual set $O \subset C^\infty(U, \mathbb{R})$ such that the following condition holds: For any $\tilde{F} \in O$ and $(0, y_0, t_0, u_0) \in U$, the function germ $F(x, y, t, u) \in \mathfrak{M}(r; k+1+n)$ given by $F(x, y, t, u) = \tilde{F}(x, y+y_0, t+t_0, u+u_0) - \tilde{F}(0, y_0, t_0, u_0)$ is a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of $F|_{t=0}$.*

2.3 Contact regular r-cubic configurations

We review a result given in [8]. Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions in n-variables which may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system $(q, z, p) = (q_1, \ldots, q_n, z, p_1, \ldots, p_n)$, where q be a coordinate system of $\mathbb{R}^n$. We equip the contact structure on $J^1(\mathbb{R}^n, \mathbb{R})$ defined by the canonical 1-form $\theta = dz - \sum_{i=1}^n p_i dq_i$. We have a natural projection $\pi: J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ by $\pi(q, z, p) = (q, z)$.

Definition 2.5 Let $w \in J^1(\mathbb{R}^n, \mathbb{R})$ and $\{L_\sigma\}_{\sigma \subset I_r}$ be a family of 2^r Legendrian submanifold germs on $(J^1(\mathbb{R}^n, \mathbb{R}), w)$. Then $\{L_\sigma\}_{\sigma \subset I_r}$ is called *a contact regular r-cubic configuration* on $J^1(\mathbb{R}^n, \mathbb{R})$ if there exists a contact embedding germ $C: (J^1(\mathbb{R}^n, \mathbb{R}), 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), w)$ such that $L_\sigma = C(L_\sigma^0)$ for all $\sigma \subset I_r$.

A function germ $F(x, y, q, z) \in \mathfrak{M}(r; k+n+1)$ is called *C-non-degenerate* if $\frac{\partial F}{\partial x}(0) = \frac{\partial F}{\partial y}(0) = 0$ and $x, F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are independent on $(\mathbb{H}^r \times \mathbb{R}^{k+n+1}, 0)$

Definition 2.6 Let $\{L_\sigma\}_{\sigma \subset I_r}$ be a contact regular r-cubic configuration on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$. Then a function germ $F(x, y, q, z) \in \mathfrak{M}(r; k+n+1)$ is called *a generating family* of $\{L_\sigma\}_{\sigma \subset I_r}$ if the following conditions hold:
(1) F is C-non-degenerate,
(2) For each $\sigma \subset I_r$, the function germ $F|_{x_\sigma = 0}$ is a generating family of L_σ, that is

$$L_\sigma = \{(q, z, \frac{\partial F}{\partial q}/(-\frac{\partial F}{\partial z})) \in (J^1(\mathbb{R}^n, \mathbb{R}), 0) | x_\sigma = \frac{\partial F}{\partial x_{I_r - \sigma}} = \frac{\partial F}{\partial y} = F = 0, x_{I_r - \sigma} \geq 0\}.$$

We say that contact regular r-cubic configurations $\{L_\sigma^1\}_{\sigma \subset I_r}$ and $\{L_\sigma^2\}_{\sigma \subset I_r}$ on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ are *Legendrian equivalent* if there exists a Legendrian equivalence Θ of π such that $L_\sigma^2 = \Theta(L_\sigma^1)$ for all $\sigma \subset I_r$.

We say that function germs $F(x, y_1, \cdots, y_{k_1}, q, z) \in \mathfrak{M}(r; k_1 + n + 1)$ and $G(x, y_1, \cdots, y_{k_2}, q, z) \in \mathfrak{M}(r; k_2 + n + 1)$ are *stably reticular $\mathcal{P}$-$\mathcal{K}$-equivalent* if F and G are reticular $\mathcal{P}$-$\mathcal{K}$-equivalent after additions of non-degenerate quadratic forms in the variables y. We also define other stably equivalences for function germs analogously.

Theorem 2.7 (Cf., [8, Theorem 5.6]) (1) *For any contact regular r-cubic configuration $\{L_\sigma\}_{\sigma \subset I_r}$ on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$, there exists a function germ $F \in \mathfrak{M}(r; k+n+1)$ which is a generating family of $\{L_\sigma\}_{\sigma \subset I_r}$.*
(2) *For any C-non-degenerate function germ $F \in \mathfrak{M}(r; k+n+1)$, there exists a contact regular r-cubic configuration of which F is a generating family.*
(3) *Two contact regular r-cubic configurations are Legendrian equivalent if and only if their generating families are stably reticular $\mathcal{P}$-$\mathcal{K}$-equivalent.*

3   Reticular Legendrian unfoldings

We consider the big 1-jet bundle $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and the canonical 1-form Θ on that space. Let $(t,q)=(t,q_1,\ldots,q_n)$ be the canonical coordinate system on $\mathbb{R}\times\mathbb{R}^n$ and $(t,q,z,s,p)=(t,q_1,\ldots,q_n,z,s,p_1,\ldots,p_n)$ be the corresponding coordinate system on $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. Then the canonical 1-form Θ is given by

$$\Theta = dz - \sum_{i=1}^{n} p_i dq_i - sdt = \theta - sdt.$$

We recall that our purpose is the investigation of one-parameter families of contact regular r-cubic configurations on $J^1(\mathbb{R}^n,\mathbb{R})$ which defined by one-parameter families of contact embedding gems $(J^1(\mathbb{R}^n,\mathbb{R}),0)\to J^1(\mathbb{R}^n,\mathbb{R})$ depending smoothly on $t\in(\mathbb{R},0)$.

Let $\{L_{\sigma,t}\}_{\sigma\subset I_r,t\in(\mathbb{R},0)}$ be a family of contact regular r-cubic configurations on $J^1(\mathbb{R}^n,\mathbb{R})$ defined by a family of contact embedding germs $C_t:(J^1(\mathbb{R}^n,\mathbb{R}),0)\to J^1(\mathbb{R}^n,\mathbb{R})$ depending smoothly on $t\in(\mathbb{R},0)$ such that $C_0(0)=0$ and $L_{\sigma,t}=C_t(L_\sigma^0)$ for all $\sigma\subset I_r$ and $t\in(\mathbb{R},0)$.

We consider the contact diffeomorphism germ C on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ as the following:

Lemma 3.1 *For any family of contact embedding germs $C_t:(J^1(\mathbb{R}^n,\mathbb{R}),0)\to J^1(\mathbb{R}^n,\mathbb{R})$ $(C_0(0)=0)$ depending smoothly on $t\in(\mathbb{R},0)$, there exists a unique function germ h on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ such that the map germ $C:(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\to(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ defined by*

$$C(t,q,z,s,p)=(t,q\circ C_t(q,z,p),z\circ C_t(q,z,p),h(t,q,z,s,p),p\circ C_t(q,z,p))$$

is a contact diffeomorphism.

Proof. We denote that $C_t(q,z,p)=(q_t(q,z,p),z_t(z,q,p),p_t(q,z,p))$. Since C_t is a contact embedding germ for all $t\in(\mathbb{R},0)$, there exists a function germ $\alpha(t,q,z,p)$ around zero with $\alpha(0)\neq 0$ such that $dz_t(z,q,p)-p_t(q,z,p)dq_t(q,\,z,p)=\alpha(t,q,z,p)(dz-pdq)$ for all fixed t. By the direct calculation of this equation, we have that

$$\frac{\partial z_t}{\partial z} - p_t\frac{\partial q_t}{\partial z} = \alpha,\ \frac{\partial z_t}{\partial q} - p_t\frac{\partial q_t}{\partial q} = -p\alpha,\ \frac{\partial z_t}{\partial p} - p_t\frac{\partial q_t}{\partial p} = 0.$$

We also calculate $C^*(dz-pdq-sdt)$ by considering the above relations. Then we have that

$$C^*(dz-pdq-sdt)$$
$$= dz_t(z,q,p)-p_t(q,z,p)dq_t(q,z,p)-h(t,q,z,s,p)dt$$
$$= \alpha(t,z,q,p)dz - \alpha(t,z,q,p)dq - (\frac{\partial z_t}{\partial t}(q,z,p)-p_t(q,z,p)\frac{\partial q_t}{\partial t}(q,z,p)-h(t,q,z,s,p))dt.$$

To make C a contact embedding, the function $h(t,q,z,s,p)$ is uniquely determined that:

$$h(t,q,z,s,p)=\frac{\partial z_t}{\partial t}(q,z,p)-p_t(q,z,p)\frac{\partial q_t}{\partial t}(q,z,p)+\alpha(t,q,z,p)s.$$

$\square$

Definition 3.2 *Let C be a contact diffeomorphism germ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$. We say that C is a $\mathcal{P}$-contact diffeomorphism if C has the form:*

$$C(t, q, z, s, p) = (t, q_C(t, q, z, p), z_C(t, q, z, p), h_C(t, q, z, s, p), p_C(t, q, z, p)). \quad (3.4)$$

We remark that a $\mathcal{P}$-contact diffeomorphism and the corresponding one-parameter family of contact embedding germs are uniquely defined by each other.

We define that $\tilde{L}_\sigma^0 = \{(t, q, z, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) | q_\sigma = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = s = z = 0, q_{I_r - \sigma} \geq 0\}$ for $\sigma \subset I_r$ and $\mathbb{L} = \{(t, q, z, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) | q_1 p_1 = \cdots = q_r p_r = q_{r+1} = \cdots = q_n = s = z = 0, q_{I_r} \geq 0\}$ be a representative as a germ of the union of $\tilde{L}_\sigma^0$ for all $\sigma \subset I_r$.

Definition 3.3 We say that a map germ $\mathcal{L} : (\mathbb{L}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ is *a reticular Legendrian unfolding* if $\mathcal{L}$ is the restriction of a $\mathcal{P}$-contact diffeomorphism. We call $\{\mathcal{L}(\tilde{L}_\sigma^0)\}_{\sigma \subset I_r}$ *the unfolded contact regular r-cubic configuration of $\mathcal{L}$.*

We note that: Let $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ be an unfolded contact regular r-cubic configuration associated with a one-parameter family of contact regular r-cubic configurations $\{L_{\sigma,t}\}_{\sigma \subset I_r, t \in (\mathbb{R}, 0)}$. Then there is the following relation between the wavefront $W_\sigma = \Pi(\tilde{L}_\sigma)$ and the family of wavefronts $W_{\sigma,t} = \pi(L_{\sigma,t})$:

$$W_\sigma = \bigcup_{t \in (\mathbb{R}, 0)} \{t\} \times W_{\sigma,t} \qquad \text{for all } \sigma \subset I_r.$$

In order to study bifurcations of wavefronts of unfolded contact regular r-cubic configurations we introduce the following equivalence relation. Let K, Ψ be contact diffeomorphism germs on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$. We say that K is *a $\mathcal{P}$-Legendrian equivalence* if K has the form:

$$K(t, q, z, s, p) = (\phi_1(t), \phi_2(t, q, z), \phi_3(t, q, z), \phi_4(t, q, z, s, p), \phi_5(t, q, z, s, p)). \quad (3.5)$$

We say that Ψ is *a reticular $\mathcal{P}$-diffeomorphism* if $\pi_t \circ \Psi$ depends only on t and Ψ preserves $\tilde{L}_\sigma^0$ for all $\sigma \subset I_r$.

Let $\{\tilde{L}_\sigma^i\}_{\sigma \subset I_r} (i = 1, 2)$ be unfolded contact regular r-cubic configurations on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$. We say that they are *$\mathcal{P}$-Legendrian equivalent* if there exist a $\mathcal{P}$-contact diffeomorphism germ K such that $\tilde{L}_\sigma^2 = K(\tilde{L}_\sigma^1)$ for all $\sigma \subset I_r$.

In order to understand the meaning of $\mathcal{P}$-Legendrian equivalence, we observe the follow: Let $\{\tilde{L}_\sigma^i\}_{\sigma \subset I_r} (i = 1, 2)$ be unfolded contact regular r-cubic configurations on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and $\{L_{\sigma,t}^i\}_{\sigma \subset I_r, t \in (\mathbb{R}, 0)}$ be the corresponding one-parameter families of contact regular r-cubic configurations on $J^1(\mathbb{R}^n, \mathbb{R})$ respectively. We take the smooth path germs $w_i : (\mathbb{R}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)$ such that $\{L_{\sigma,t}^i\}_{\sigma \subset I_r}$ are defined at $w_i(t)$ for $i = 1, 2$. Suppose that there exists a $\mathcal{P}$-Legendrian equivalence K from $\{\tilde{L}_\sigma^1\}_{\sigma \subset I_r}$ to $\{\tilde{L}_\sigma^2\}_{\sigma \subset I_r}$ of the form (3.5). We set $W_{\sigma,t}^i$ be the wavefront of $L_{\sigma,t}^i$ for $\sigma \subset I_r$, $t \in (\mathbb{R}, 0)$ and $i = 1, 2$. We define the family of diffeomorphism $g_t : (\mathbb{R}^n \times \mathbb{R}, \pi(w_1(t))) \to (\mathbb{R}^n \times \mathbb{R}, \pi(w_2(t)))$ by $g_t(q, z) = (\phi_2(t, q, z), \phi_3(t, q, z))$. Then we have that $g_t(W_{\sigma,t}^1) = W_{\sigma,\phi_1(t)}^1$ for all $\sigma \subset I_r$, $t \in (\mathbb{R}, 0)$.

We also define the equivalence relation among reticular Legendrian unfoldings. Let $\mathcal{L}_i : (\mathbb{L}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0), (i = 1, 2)$ be reticular Legendrian unfoldings. We say that $\mathcal{L}_1$ and

$\mathcal{L}_2$ are $\mathcal{P}$-*Legendrian equivalent* if there exist a $\mathcal{P}$-contact diffeomorphism germ K and a reticular $\mathcal{P}$-diffeomorphism Ψ such that $K \circ \mathcal{L}_1 = \mathcal{L}_2 \circ \Psi$.

We remark that two reticular Legendrian unfoldings are $\mathcal{P}$-Legendrian equivalent if and only if the corresponding unfolded contact regular r-cubic configurations are $\mathcal{P}$-Legendrian equivalent.

By the same proof of Lemma 5.3 in [8], we have the follow:

Lemma 3.4 *Let $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ be an unfolded contact regular r-cubic configuration on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$. Then there exists a $\mathcal{P}$-contact diffeomorphism germ C on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that C defines $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ and C preserves the canonical 1-form.*

By this lemma we may assume that all reticular Legendrian unfoldings (and all unfolded contact regular r-cubic configurations) are defined by $\mathcal{P}$-contact diffeomorphism germs which preserve the canonical 1-form.

We can construct generating families of reticular Legendrian unfoldings. A function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; k+1+n+1)$ is called $\mathcal{P}$-*C-non-degenerate* if $\frac{\partial F}{\partial x}(0) = \frac{\partial F}{\partial y}(0) = 0$ and $x, t, F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are independent on $(\mathbb{H}^k \times \mathbb{R}^{k+1+n+1}, 0)$.

A $\mathcal{P}$-C-non-degenerate function germ $F(x, y, t, q, z) \in \mathfrak{M}(r; k+1+n+1)$ is called *a generating family* of a reticular Legendrian unfoldings $\mathcal{L}$ if

$$\mathcal{L}(\tilde{L}_\sigma^0) = \{(t, q, z, \frac{\partial F}{\partial t}/(-\frac{\partial F}{\partial z}), \frac{\partial F}{\partial q}/(-\frac{\partial F}{\partial z})) \in (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)|$$

$$x_\sigma = F = \frac{\partial F}{\partial x_{I_r - \sigma}} = \frac{\partial F}{\partial y} = 0, x_{I_r - \sigma} \geq 0\} \text{ for all } \sigma \subset I_r.$$

We remark that for a $\mathcal{P}$-C-non-degenerate function germ $F(x, y, t, q, z)$, the function germ $F(\cdot, \cdot, t, \cdot, \cdot)$ is C-non-degenerate.

Lemma 3.5 *Let C be a $\mathcal{P}$-contact diffeomorphism germ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ which preserves the canonical 1-form. If the map germ*

$$(T, Q, Z, S, P) \rightarrow (T, Q, Z, s_C(T, Q, Z, S, P), p_C(T, Q, Z, S, P))$$

is diffeomorphism at 0. Then there exists a function germ $H(T, Q, p) \in \mathfrak{M}(1+n+n)^2$ such that the canonical relation P_C associated with C has the form:

$$P_C = \{(T, Q, Z, -\frac{\partial H}{\partial T}(T, Q, p) + s, -\frac{\partial H}{\partial Q}, T, -\frac{\partial H}{\partial p}, H - \langle \frac{\partial H}{\partial p}, p \rangle + Z, s, p)\}, \qquad (3.6)$$

and the function germ $F \in \mathfrak{M}(r; n+1+n+1)$ defined by $F(x, y, t, q, z) = -z + H(t, x, 0, y) + \langle y, q \rangle$ is a generating family of the reticular Legendrian unfolding $C|_\mathbb{L}$.

Proof. We have that $dz - sdt - pdq = dZ - SdT - PdQ$ on P_C. It follows that $d(z - Z) = sdt + pdq - SdT - PdQ$ and $d(z - Z + st + pq) = -tds - qdp - SdT - PdQ$. Then there exists a function germ $H'(T, Q, s, p) \in \mathfrak{M}(1+n+1+n)^2$ such that

$$z - Z - st - pq = H'(T, Q, s, p), \ t = -\frac{\partial H'}{\partial s}, \ q = -\frac{\partial H'}{\partial p}, S = -\frac{\partial H'}{\partial T}, \ P = -\frac{\partial H'}{\partial Q} \text{ on } P_C.$$

Since $t = T = -\frac{\partial H'}{\partial s}$ on P_C, we have that $H'(T,Q,s,p) = H(T,Q,p) - Ts$ for some $H(T,Q,p) \in \mathfrak{M}(1+n+n)^2$. Then

$$z - Z - Ts - \langle -\frac{\partial H}{\partial p}, p \rangle = H(T,Q,p) - Ts.$$

Therefore we have that

$$z = H(T,Q,p) - \langle \frac{\partial H}{\partial p}, p \rangle + Z$$

and have the required form of P_C. By the direct calculation with the form P_C, we have that F is a generating family of $C|_{\mathbb{L}}$. $\qquad\square$

We have the following theorem which gives the relations between reticular Legendrian unfoldings and their generating families.

Theorem 3.6 (1) *For any reticular Legendrian unfolding* $\mathcal{L} : (\mathbb{L},0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$, *there exists a function germ* $F(x,y,t,q,z) \in \mathfrak{M}(r; k+1+n+1)$ *which is a generating family of* $\mathcal{L}$.
(2) *For any* $\mathcal{P}$-C-*non-degenerate function germ* $F(x,y,t,q,z) \in \mathfrak{M}(r; k+1+n+1)$ *with* $\frac{\partial F}{\partial t} = \frac{\partial F}{\partial q} = 0$, *there exists a reticular Legendrian unfolding* $\mathcal{L} : (\mathbb{L},0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ *of which* F *is a generating family.*
(3) *Two reticular Legendrian unfolding are* $\mathcal{P}$-*Legendrian equivalent if and only if their generating families are stably reticular* t-$\mathcal{P}$-$\mathcal{K}$-*equivalent.*

This theorem is proved by analogously methods of [7], [8]. We give a sketch of the proof.
(1) Let C be a $\mathcal{P}$-contact diffeomorphism germ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that $C|_{\mathbb{L}} = \mathcal{L}$. We may assume that $C^*\Theta = \Theta$. By taking a $\mathcal{P}$-Legendrian equivalence of $\mathcal{L}$, we may assume that the canonical relation P_C associated with C has the form (3.6) for the function germ $H \in \mathfrak{M}(1+n+n)^2$. Then the function germ $F(x,y,t,q,z) \in \mathfrak{M}(r; n+1+n+1)$ defined by

$$F(x,y,t,q,z) = -z + H(t,x_1,\ldots,x_r,0,y) + \langle y, q \rangle$$

is a generating family of $\mathcal{L}$.
(2) Let a $\mathcal{P}$-C-non-degenerate function germ $F(x,y,t,q,z) \in \mathfrak{M}(r; k+1+n+1)$ with $\frac{\partial F}{\partial t} = \frac{\partial F}{\partial q} = 0$ be given. By [8, Lemma 2.1], we may assume that F has the form $F(x,y,t,q,z) = -z + F_0(x,y,t,q)$ for some $F_0 \in \mathfrak{M}(r; k+1+n)$. Choose an $(n-r) \times k$-matrix A and an $(n-r) \times n$-matrix B such that the matrix

$$\begin{pmatrix} \frac{\partial^2 F_0}{\partial x \partial y} & \frac{\partial^2 F_0}{\partial x \partial q} & \frac{\partial^2 F_0}{\partial x \partial t} \\ \frac{\partial^2 F_0}{\partial y \partial y} & \frac{\partial^2 F_0}{\partial y \partial q} & \frac{\partial^2 F_0}{\partial y \partial t} \\ A & B & 0 \\ 0 & 0 & 1 \end{pmatrix}_0 \quad \text{is invertible.} \qquad (3.7)$$

Let $F' \in \mathfrak{M}(r+k+1+n+1)$ be a function germ which is obtained by an extension the source space of F to $(\mathbb{R}^{r+k+1+n+1}, 0)$. Define the function $G(S,Q,y,t,q,z) \in \mathfrak{M}(n+1+1+k+1+n+1)$ by that

$$G(Q,Z,S,y,t,q,z) = -z + F'(Q_1,\ldots,Q_r,y,t,q) +$$

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$$(Q_{r+1},\ldots,Q_n)A\begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} + (Q_{r+1},\ldots,Q_n)B\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} + St.$$

Then G is a generating family of the canonical relation P_C associated with some $\mathcal{P}$-contact diffeomorphism germ C. The function germ F is a generating family of the reticular Legendrian unfolding $C|_{\mathbb{L}}$.

(3) We need only to prove that: *If $F_1, F_2 \in \mathfrak{M}(r;k+1+n)$ are generating families of the same reticular Legendrian unfolding, then they are reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent.*

We may reduce that F_i has the form $F_i(x,y,t,q,z) = -z + F_i^0(x,y,t,q)$ for $F_i^0 \in \mathfrak{M}(r;k+1+n)$, $i=1,2$. Then F_1^0 and F_2^0 are generating families of the same reticular Lagrangian map in the sense of [7]. By [7, p.587 the assertion (3)], there exists a reticular $\mathcal{R}$-equivalence from F_2^0 to F_1^0 of the form:

$$F_1^0(x,y,t,q) = F_2^0(x\phi_1(x,y,t,q), \phi_2(x,y,t,q),t,q).$$

This means that F_1 and F_2 are reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent.

4   Stabilities

In this section we shall define several stabilities of reticular Legendrian unfoldings and prove that they and the stabilities of corresponding generating families are all equivalent.

Let U be an open set in $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. We consider contact diffeomorphism germs on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ and contact embeddings from U to $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. Let (T,Q,S,Z,P) and (t,q,z,s,p) be canonical coordinates of the source space and the target space respectively. We define the following notations:

$\imath : (J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})\cap\{Z=0\},0) \to (J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ be the inclusion map on the source space,

$$C_T(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0) = \{C|C \text{ is a } \mathcal{P}\text{-contact diffeomorphism germ}$$
$$\text{on } (J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\},$$
$$C_T^\Theta(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0) = \{C \in C_T(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)| \ C^*\Theta = \Theta\},$$
$$C_T^Z(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0) = \{C\circ\imath \ |C \in C_T(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\},$$
$$C_T^{\Theta,Z}(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0) = \{C\circ\imath \ |C \in C_T^\Theta(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\}.$$

Let $V = U\cap\{Z=0\}$ and $\tilde{\imath} : V \to U$ be the inclusion map.

$$C_T(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})) = \{\tilde{C} : U \to J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})|$$
$$\tilde{C} \text{ is a contact embedding of the form } (3.4)\},$$
$$C_T^\Theta(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})) = \{\tilde{C} \in C_T(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})) \ |\tilde{C}^*\Theta = \Theta\},$$
$$C_T^Z(V,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})) = \{\tilde{C}\circ\tilde{\imath} \ |\tilde{C} \in C_T(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))\},$$
$$C_T^{\Theta,Z}(V,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})) = \{\tilde{C}\circ\tilde{\imath} \ |\tilde{C} \in C_T^\Theta(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))\}.$$

Definition 4.1 Stability: We say that a reticular Legendrian unfolding $\mathcal{L}$ is *stable* if the following condition holds: Let $C_0 \in C_T(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0))$ be a $\mathcal{P}$-contact diffeomorphism germ such that $C_0|_{\mathbb{L}} = \mathcal{L}$ and $\tilde{C}_0 \in C_T(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))$ be a representative of C_0. Then there exists an open neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ in C^∞-topology such that for any $\tilde{C} \in N_{\tilde{C}_0}$,

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there exists a point $w_0 = (T^0, 0, \ldots, 0, P^0_{r+1}, \ldots, P^0_n) \in U$ such that $\mathcal{L}'_{w_0}$ and $\mathcal{L}$ are $\mathcal{P}$-Legendrian equivalent, where a reticular Legendrian unfolding $\mathcal{L}'_{w_0}$ is chosen that the reticular Legendrian unfolding $\tilde{C}|_{\mathbb{L}} : (\mathbb{L}, w_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), \tilde{C}(w_0))$ and $\mathcal{L}'_{w_0}$ are $\mathcal{P}$-Legendrian equivalent. We remark that the definition of stability is not depend on a choice of $\mathcal{L}'_{w_0}$.

Homotopically stability: A one-parameter family of $\mathcal{P}$-contact diffeomorphism germs $\bar{C} : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \times \mathbb{R}, (0,0)) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)((T,Q,Z,S,P,\tau) \mapsto C_\tau(T,Q,Z,S,P))$ is called a *reticular $\mathcal{P}$-contact deformation* of $\mathcal{L}$ if $C_0|_{\mathbb{L}} = \mathcal{L}$. A map germ $\bar{\Psi} : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \times \mathbb{R}, (0,0)) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)((T,Q,Z,S,P,\tau) \mapsto \Psi_\tau(T,Q,Z,S,P))$ is called a *one-parameter deformation of reticular $\mathcal{P}$-diffeomorphisms* if $\Psi_0 = id_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}$ and Ψ_t is a reticular $\mathcal{P}$-diffeomorphism for all t around 0. We say that a reticular Legendrian unfolding $\mathcal{L}$ is *homotopically stable* if for any reticular $\mathcal{P}$-Legendrian deformation $\bar{C} = \{C_\tau\}$ of $\mathcal{L}$, there exist one-parameter family of $\mathcal{P}$-Legendrian equivalences $\bar{K} = \{K_\tau\}$ with $K_0 = id_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}$ and a one-parameter deformation of reticular $\mathcal{P}$-diffeomorphisms $\bar{\Psi} = \{\Psi_\tau\}$ such that $C_\tau = K_\tau \circ C_0 \circ \Psi_\tau$ for t around 0.

Infinitesimally stability: Let $C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be a $\mathcal{P}$-contact diffeomorphism germ. We say that a vector field v on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ along C is *an infinitesimal $\mathcal{P}$-contact transformation* of C if there exists a reticular $\mathcal{P}$-Legendrian deformation $\bar{C} = \{C_\tau\}$ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that $C_0 = C$ and $\frac{dC_\tau}{d\tau}|_{\tau=0} = v$. We say that a vector field ξ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ is *infinitesimal reticular $\mathcal{P}$-diffeomorphism* if there exists a one-parameter deformation of reticular $\mathcal{P}$-diffeomorphisms $\bar{\Psi} = \{\Psi_\tau\}$ such that $\frac{d\Psi_\tau}{d\tau}|_{\tau=0} = \xi$. We say that a vector field η on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ is *infinitesimal $\mathcal{P}$-Legendrian equivalence* if there exists a one-parameter family of $\mathcal{P}$-Legendrian equivalences $\bar{K} = \{K_\tau\}$ such that $K_0 = id_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}$ and $\frac{dK_\tau}{d\tau}|_{\tau=0} = \eta$. We say that a reticular Legendrian unfolding $\mathcal{L}$ is *infinitesimally stable* if for any extension C of $\mathcal{L}$ and any infinitesimal $\mathcal{P}$-contact transformation v of C, there exists an infinitesimal reticular $\mathcal{P}$-diffeomorphism ξ and an infinitesimal $\mathcal{P}$-Legendrian equivalence η such that $v = C_* \xi + \eta \circ C$.

We may take an extension of a reticular Legendrian unfolding $\mathcal{L}$ by an element of $C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ by Lemma 3.4. Then as the remark after the definition of the stability of reticular Legendrian maps in [8, p.121], we may consider the following other definitions of stabilities of reticular Legendrian unfoldings: (1) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$ and $C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ to $C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$ and $C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ of original definition respectively. (2) The definition given by replacing to $C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$ and $C_T^Z(V, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ respectively. (3) The definition given by replacing to $C_T^{\Theta,Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$ and $C_T^{\Theta,Z}(V, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ respectively, where $V = U \cap \{Z = 0\}$.

Then we have the following Lemma which is proved by the same method of the proof of Lemma 7.2 in [8]

Lemma 4.2 *The original definition and other three definitions of stabilities of reticular Legendrian unfoldings are all equivalent.*

By This lemma, we may choose an extension of a reticular Legendrian unfolding from among all of $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$, $C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$, $C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$, and $C_T^{\Theta,Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0))$.

We say that a function germ H on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ is $\mathcal{P}$-*fiber preserving* if H has the form $H(t,q,z,s,p)=\sum_{j=1}^n h_j(t,q,z)p_j+h_0(t,q,z)+a(t)s$.

Lemma 4.3 *Let $C\in C_T(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$. Then the followings hold:* (1) *A vector field germ v on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ along C is an infinitesimal $\mathcal{P}$-contact transformation of C if and only if there exists a function germ f on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ such that f does not depend on s and $v=X_f\circ C$.*
(2) *A vector field germ η on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ is an infinitesimal $\mathcal{P}$-Legendrian equivalence if and only if there exists a $\mathcal{P}$-fiber preserving function germ H on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ such that $\eta=X_H$.*
(3) *A vector field ξ on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ is an infinitesimal reticular $\mathcal{P}$-diffeomorphism if and only if there exists a function germ $g\in B$ such that $\xi=X_g$, where $B=\langle q_1p_1,\ldots,q_rp_r,q_{r+1},\ldots,q_n,z\rangle_{\mathcal{E}_{t,q,z,p}}+\langle s\rangle_{\mathcal{E}_t}$*

Proof. A vector field X on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ is a contact Hamiltonian vector field if and only there exists a function germ f on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ such that $X=X_f$, that is

$$X=\sum_{i=1}^n\left(\frac{\partial f}{\partial q_i}-p_i\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i}+\left(\frac{\partial f}{\partial t}+s\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial s}$$
$$-\sum_{i=1}^n\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial f}{\partial s}\frac{\partial}{\partial t}+\left(H-\sum_{i=1}^n p_i\frac{\partial f}{\partial p_i}-s\frac{\partial f}{\partial s}\right)\frac{\partial}{\partial z}.$$

(1) A vector field v is an infinitesimal $\mathcal{P}$-contact transformation of C if and only if $v=H_f\circ C$ and $\frac{\partial f}{\partial s}=0$. This holds if and only if $v=H_f\circ C$ and f does not depend on s.
(2) A vector field germ η is an infinitesimal $\mathcal{P}$-Legendrian equivalence if and only if there exists a fiber preserving function germ H such that $\eta=X_H$ by and $\frac{\partial H}{\partial s}=a(t)$ for some function germ $a(t)$. This holds if and only if $\eta=X_H$ and H is a $\mathcal{P}$-fiber preserving function.
(3) A vector field ξ is an infinitesimal reticular $\mathcal{P}$-diffeomorphism of C if and only if there exists a function germ $g\in\langle q_1p_1,\ldots,q_rp_r,q_{r+1},\ldots,q_n,z,s\rangle_{\mathcal{E}_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}}$ such that $\xi=X_g$ since X_g is tangent to $\tilde{L}_\sigma^0$ for all $\sigma\subset I_r$, and $\frac{\partial g}{\partial s}=a(t)$, this holds if and only if $\xi=X_g$ and $g\in B$. $\qquad\square$

Let U be a neighbourhood of 0 in $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. We define:

$$J^l_{C_T^\Theta}(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))=\{j^lC(w_0)\in J^l(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))|$$
$$C:(U,w_0)\to J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})\text{ is a }\mathcal{P}\text{-contact embedding germ which preserves }\Theta\}.$$

Theorem 4.4 ($\mathcal{P}$-Contact transversality theorem) *Let $Q_i,i=1,2,\ldots$ are submanifolds of $J^l_{C_T^\Theta}(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))$. Then the set*

$$T=\{C\in C_T^\Theta(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))|j^lC\text{ is transversal to }Q_i\text{ for all }i\in\mathbb{N}\}$$

is a residual set in $C_T^\Theta(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))$

Proof. We reduce our assertion to local situations by choosing a countable covering of Q_i by sufficiently small compact sets $K_{i,j}$'s. We fix a $\mathcal{P}$-contact embedding $C\in C_T^\Theta(U,J^1(\mathbb{R}\times$

$\mathbb{R}^n, \mathbb{R}$)). For each $w_0 \in U$ there exist local contact coordinate systems of U around w_0 and $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ around $C(w_0)$ such that C is given by $(t,q,z,s,p) \mapsto (t,q,z,s,p)$ around 0.

For each i,j we take E by a sufficiently small neighbourhood of 0 in $P(2n+3,1,l+1)$ and take a smooth function $\rho: U \to [0,1]$ such that ρ is equal to 1 in a neighbourhood of w_0 and 0 outside a compact set, where $P(2n+3,1,l+1)$ is the set of polynomials on $(2n+3)$-variables with degree $\leq l+1$.

For each $H \in E$ we define $H'(T,Q,Z,s,p) = \rho(T,Q,Z,s,p)H(T,Q,Z,s,p) - \langle Q,p \rangle - Ts$. and $\psi_H(T,Q,Z,s,p) = (T,Q,Z, -\frac{\partial H'}{\partial T}(T,Q,Z,s,p), -\frac{\partial H'}{\partial Q}(T,Q,Z,s,p))$.

Then there exists a neighbourhood U' of 0 such that ψ_H is an embedding on U' and identity outside a compact set for any $H \in E$. Therefore there exists a neighbourhood W of 0 in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ such that the map $E \to C^\infty(W,U')$, $H \mapsto (\psi_H^{-1})|_W$ is well defined and continuous. Each $(\psi_H^{-1})|_W$ is equal the identity map outside a compact set. We set

$$\phi(H)(T,Q,Z,S,P) = (-\frac{\partial H'}{\partial s}, -\frac{\partial H'}{\partial p}, H' - \langle \frac{\partial H'}{\partial p}, p \rangle + Z, s, p) \circ (\psi_{H'}^{-1})(T,Q,Z,S,P)$$

for $(T,Q,Z,S,P) \in W$. Then $\phi(H)$ is a $\mathcal{P}$-contact embedding germ preserving Θ around 0 which has the canonical relation with the generating function $H(T,Q,Z,s,p)$ and equal to C outside a compact set. It follows that the source space of $\phi(H)$ can be extended naturally to $E \times U$. We denote this by $\phi_H \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$. Then the map

$$\Phi: E \times U \to J_{C_T^\Theta}^l(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})), \quad \Phi(H,w) = j^l(\phi_H)(w)$$

is a submersion around $(0,w_0)$, hence is transversal to $K_{i,j}$. So we have the result. $\square$

We denote the ring $\mathcal{E}(1+n+n)$ on the coordinates (t,q,p) by $\mathcal{E}_{t,q,p}$ and denote other notations analogously.

Theorem 4.5 *Let $\mathcal{L}$ be a reticular Legendrian unfolding with a generating family $F(x,y,t,q,z)$. Then the following are equivalent.*
(u) F is a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of $F|_{t=0}$.
(hs) $\mathcal{L}$ is homotopically stable.
(is) $\mathcal{L}$ is infinitesimally stable.
(a) $\mathcal{E}_{t,q,p} = B_0 + \langle 1, p_1 \circ C', \ldots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} + \langle s \circ C' \rangle_{\mathcal{E}_t}$, where $C' = C|_{z=s=0}$ and $B_0 = \langle q_1 p_1, \ldots, q_r p_r, q_{r+1}, \ldots, q_n \rangle_{\mathcal{E}_{t,q,p}}$.

We remark that sufficiently near reticular Legendrian unfoldings of stable one are stable by the condition (a).

Proof. (u)$\Rightarrow$(hs): Let a reticular $\mathcal{P}$-Legendrian deformation $\bar{C} = \{C_\tau\}$ of $\mathcal{L}$ be given. The homotopically stability of reticular Legendrian unfoldings is invariant under $\mathcal{P}$-Legendrian equivalences, we may assume that the map germs

$$(T,Q,Z,S,P) \to (T,Q,Z,s \circ C_\tau(T,Q,P), p \circ C_\tau(T,Q,P))$$

are a diffeomorphism at 0 hence for all τ. By Lemma 3.5, there exists a one-parameter family $H_\tau(T,Q,p) \in \mathfrak{M}(1+n+n)^2$ depending smoothly on $\tau \in (\mathbb{R},0)$ such that the canonical relations P_{C_τ} associated with C_τ has the form:

$$P_{C_\tau} = \{(T,Q,Z, -\frac{\partial H_\tau}{\partial T}(T,Q,p) + s, -\frac{\partial H_\tau}{\partial Q}, T, -\frac{\partial H_\tau}{\partial p}, H_\tau - \langle \frac{\partial H_\tau}{\partial p}, p \rangle + Z, s, p)\}.$$

Then the function germ $F_\tau \in \mathfrak{M}(r; n+1+n+1)$ defined by

$$F_\tau(x,y,t,q,z) = -z + H_\tau(t,x,0,y) + \langle y,q \rangle$$

are generating families of reticular Legendrian unfoldings $\mathcal{L}_\tau := C_\tau|_{\mathbb{L}}$ for $\tau \in (\mathbb{R},0)$. Since F_0 is a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of $F|_{t=0}$, it follows that F_0 is a reticular t-$\mathcal{P}$-$\mathcal{K}$-homotopically stable unfolding of $F|_{t=0}$ by Theorem 2.3. Therefore there exists a one-parameter family of reticular t-$\mathcal{P}$-$\mathcal{K}$-isomorphism from F_τ to F_0 depending smoothly on τ. This means that there exists a one-parameter family of $\mathcal{P}$-Legendrian equivalences K_τ depending smoothly on τ such that

$$C_\tau(L_\sigma^0) = K_\tau \circ \mathcal{L}(L_\sigma^0) \text{ for all } \sigma \subset I_r, \quad \tau \in (\mathbb{R},0).$$

Then the map germ $\Psi_\tau := C_0^{-1} \circ K_\tau^{-1} \circ C_\tau$ gives a one-parameter deformation of reticular $\mathcal{P}$-diffeomorphism on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and we have that $C_\tau = K_\tau \circ C_0 \circ \Psi_\tau$. This means that $\mathcal{L}$ is homotopically stable.

(hs)$\Rightarrow$(is): Let $C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be an extension of $\mathcal{L}$ and v be an infinitesimal $\mathcal{P}$-contact transformation of C. Then there exists a reticular $\mathcal{P}$-Legendrian deformation $\bar{C} = \{C_\tau\}$ of C such that $v = \frac{dC_\tau}{d\tau}|_{\tau=0}$. Then there exist a one-parameter of $\mathcal{P}$-Legendrian equivalences $\bar{K} = \{K_\tau\}$ and a one-parameter deformation of reticular $\mathcal{P}$-diffeomorphisms $\bar{\Psi} = \{\Psi_\tau\}$ such that $C_\tau = K_\tau \circ C_0 \circ \Psi_\tau$ for $\tau \in (\mathbb{R},0)$. Then we have that

$$v = \frac{dC_\tau}{d\tau}|_{\tau=0} = \frac{dK_\tau}{d\tau}|_{\tau=0} \circ C_0 + (C_0)_* \left(\frac{d\Psi_\tau}{d\tau}|_{\tau=0} \right).$$

(is)$\Rightarrow$(a): Let a function germ $f \in \mathcal{E}_{t,q,p}$ be given. We define the function germ f' on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ by $f'(t,q,z,s,p) = f \circ \pi_{T,Q,P} \circ C^{-1}(t,q,z,a(t,q,z,p),p)$, where $a(t,q,z,p) = s \circ C(t,q,z,0,p)$. Since f' does not depend on s, it follows that $X_{f'} \circ C$ is an infinitesimal $\mathcal{P}$-contact transformation of C. Therefore there exist an infinitesimal $\mathcal{P}$-Legendrian equivalence η and an infinitesimal reticular $\mathcal{P}$-diffeomorphism ξ such that $X_{f'} \circ C = C_* \xi + \eta \circ C$. By Lemma 4.3, there exist a $\mathcal{P}$-fiber preserving function germ H on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and $g \in B$ such that $\xi = X_g$ and $\eta = X_H$. Then we have that $f' \circ C = g + H \circ C$. Since $f' \circ C(T,Q,Z,S,P) = f \circ \pi_{T,Q,P} \circ C^{-1}(t,q,z,a(T,Q,Z,0,P),p) = f(T,Q,Z,0,P) = f(T,Q,P)$ and H has the form $H(t,q,z,s,p) = \sum_{i=1}^n h_i(t,q,z)p_i + h_0(t,q,z) + h'(t)s$, We have that

$$f \equiv \sum_{i=1}^n (h_i(\Pi \circ C'))(p_i \circ C') + h_0(\Pi \circ C') + (h'(t \circ C'))(s \circ C') \mod B_0.$$

Since $t \circ C = t$, we have the required form.

(a)$\Rightarrow$(u): By Lemma 3.5, there exists a function germ $H(T,Q,p) \in \mathfrak{M}(1+n+n)^2$ such that the function germ $H(T,Q,p) - Ts$ is a generating function of P_C. Then the function germ $F(x,y,t,q,z) \in \mathfrak{M}(r; n+1+n+1)$ given by $F(x,y,t,q,z) = -z + H(t,x,0,y) + \langle y,q \rangle$ is a generating family of $\mathcal{L}$. Then $P_C' := P_C|_{Z=S=0}$ has the form

$$P_C' = \{(T,Q,-\frac{\partial H}{\partial Q}, T, -\frac{\partial H}{\partial p}, H - \langle \frac{\partial H}{\partial p}, p \rangle, \frac{\partial H}{\partial T}, p)\}.$$

Then the map germ $P_C' \to (\mathbb{R}^{1+n+n}, 0)$, $w \mapsto \pi_{T,Q,P}(w)$ is a diffeomorphism. We set $C(F) = \{(x,y,t,q,z) \in (\mathbb{H}^r \times \mathbb{R}^{n+1+n+1}, 0) | F = x\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0\}$. We also define the map germ $C(F) \to P_C'$ by

$$(x,y,t,q,z) \mapsto (t,x,0,-\frac{\partial F}{\partial x}, -\frac{\partial H}{\partial Q_{r+1}}(t,x,0,y), -\frac{\partial H}{\partial Q_n}, t, q, z, \frac{\partial F}{\partial t}, y).$$

Then the composition of the above two map germs induces the map germ $\iota: \mathcal{E}_{T,Q,P}/B_0 \to \mathcal{E}_{C(F)}$. We denote T, Q, P for the variables on the source space of this map germ. Then the correspondence is given that:

$$T \mapsto t, Q_1 \mapsto x_1, \ldots, Q_r \mapsto x_r, P_1 \mapsto -\frac{\partial F}{\partial x_1}, \ldots, P_r \mapsto -\frac{\partial F}{\partial x_r},$$

$$t \circ C'(T,Q,P) \mapsto t, q \circ C'(T,Q,P) \mapsto q, z \circ C'(T,Q,P) \mapsto z,$$

$$s \circ C'(T,Q,P) \mapsto \frac{\partial F}{\partial t}, p \circ C'(T,Q,P) \mapsto y, ((\Pi \circ C')^* \mathcal{E}_{t,q,z}) \mapsto \mathcal{E}_{t,q,z}, ((t \circ C')^* \mathcal{E}_t) \mapsto \mathcal{E}_t.$$

Then (a) is transferred that

$$\mathcal{E}(r; n+1+n+1) = \langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle_{\mathcal{E}(r; n+1+n+1)}$$

$$+ \langle 1(=-\frac{\partial F}{\partial z}), y_1(=\frac{\partial F}{\partial q_1}), \ldots, y_n(=\frac{\partial F}{\partial q_n}) \rangle_{\mathcal{E}_{t,q,z}} + \langle \frac{\partial F}{\partial t} \rangle_{\mathcal{E}_t}.$$

It follows that F is a reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimal stable unfolding of $F|_{t=0}$. $\qquad \square$

5 Genericity

In order to give a generic classification of reticular Legendrian unfoldings, we reduce our investigation to finitely dimensional jet-spaces of contact diffeomorphism germs.

Definition 5.1 Let $\mathcal{L}: (\mathbb{L}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be a reticular Legendrian unfolding. We say that $\mathcal{L}$ is l-determined if the following condition holds: For any extension $C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ of $\mathcal{L}$, the reticular Legendrian unfolding $C'|_{\mathbb{L}}$ and $\mathcal{L}$ are $\mathcal{P}$-Legendrian equivalent for any $C' \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ satisfying that $j^l C(0) = j^l C'(0)$.

As Lemma 4.2, we may consider the following other definition of finitely determinacies of reticular Legendrian maps:
(1) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $C_T^{\Theta}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.
(2) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.
(3) The definition given by replacing $C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ to $C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.
Then the following holds:

Proposition 5.2 Let $\mathcal{L}: (\mathbb{L}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be a reticular Legendrian unfolding. Then
(A) If $\mathcal{L}$ is l-determined of the original definition, then $\mathcal{L}$ is l-determined of the definition (1).
(B) If $\mathcal{L}$ is l-determined of the definition (1), then $\mathcal{L}$ is l-determined of the definition (3).
(C) If $\mathcal{L}$ is l-determined of the definition (3), then $\mathcal{L}$ is $(l+1)$-determined of the definition (2).
(D) If $\mathcal{L}$ is l-determined of the definition (2), then $\mathcal{L}$ is l-determined of the original definition.

Proof. We need only to prove (C). Let $C \in C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be an extension of $\mathcal{L}$. Let $C' \in C_T^Z(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ satisfying $j^{l+1}C(0) = j^{l+1}C'(0)$ be given. Then there exist function germs $f(T,Q,S,P), g(T,Q,S,P) \in \mathcal{E}(2n+2)$ such that $C^*(dz - sdt - pdq) = -f(SdT + PdQ), C'^*(dz - sdt - pdq) = -g(SdT + PdQ)$. Indeed f is defined by that

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$fP_i = -\frac{\partial z_C}{\partial Q_i} + p_C \frac{\partial q_C}{\partial Q_i}$ for $i = 1, \ldots, n$ and $fS = -\frac{\partial z_C}{\partial T} + p_C \frac{\partial q_C}{\partial T}$. We define the diffeomorphism germs ϕ, ψ on $(J^1(\mathbb{R}^n, \mathbb{R}) \cap \{Z = 0\}, 0)$ by $\phi(T, Q, S, P) = (T, Q, fS, fP), \psi(T, Q, S, P) = (T, Q, gS, gP)$. We set $C_1 := C \circ \phi^{-1}, C_1' := C' \circ \psi^{-1} \in C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ Then $j^l \phi(0)$ and $j^l \psi(0)$ depends only on $j^{l+1} C(0)$, therefore we have that $j^l C_1(0) = j^l C_1'(0)$. Since $\mathcal{L}$ and $C_1|_{\mathbb{L}}$ are $\mathcal{P}$-Legendrian equivalent, it follows that $C_1|_{\mathbb{L}}$ and $C_1'|_{\mathbb{L}}$ are $\mathcal{P}$-Legendrian equivalent. Therefore we have that $\mathcal{L}$ and $C'|_{\mathbb{L}}$ are $\mathcal{P}$-Legendrian equivalent. $\qquad \square$

Theorem 5.3 Let $\mathcal{L} : (\mathbb{L}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be a reticular Legendrian unfolding. If $\mathcal{L}$ is infinitesimally stable then $\mathcal{L}$ is $(n+5)$-determined.

Proof. It is enough to prove $\mathcal{L}$ is $(n+4)$-determined of Definition 5.1 (3). Let $C \in C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be an extension of $\mathcal{L}$. Since the finitely determinacy of reticular Legendrian unfoldings is invariant under $\mathcal{P}$-Legendrian equivalences, we may assume that P_C has the form

$$P_C = \{(T, Q, Z, -\frac{\partial H}{\partial T}(T, Q, p) + s, -\frac{\partial H}{\partial Q}, T, -\frac{\partial H}{\partial p}, H - \langle \frac{\partial H}{\partial p}, p \rangle + Z, s, p)\}$$

for some function germ $H(T, Q, p) \in \mathfrak{M}(2n+1)^2$. Then $F(x, y, t, q, z) = -z + H_0(x, y, t) + \langle y, q \rangle \in \mathfrak{M}(r; n+1+n+1)$ is a generating family of $\mathcal{L}$, where $H_0(x, y, t) = H(t, x, 0, y) \in \mathfrak{M}(r; n+1)^2$. By Theorem 4.5, we have that F is a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of $f(x, y, q, z) := -z + H_0(x, y, 0) + \langle y, q \rangle \in \mathfrak{M}(r; n+n+1)$. This means that

$$\mathcal{E}(r; n+1+n+1) = \langle F, x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle_{\mathcal{E}(r; n+1+n+1)} + \langle 1, \frac{\partial F}{\partial q} \rangle_{\mathcal{E}(1+n+1)} + \langle \frac{\partial F}{\partial t} \rangle_{\mathcal{E}(1)}.$$

By the restriction of this to $q = z = 0$, we have that

$$\mathcal{E}(r; n+1) = \langle H_0, x\frac{\partial H_0}{\partial x}, \frac{\partial H_0}{\partial y} \rangle_{\mathcal{E}(r; n+1)} + \langle 1, y_1, \ldots, y_n, \frac{\partial H_0}{\partial t} \rangle_{\mathcal{E}(1)}. \tag{5.8}$$

This means that

$$\mathfrak{M}(r; n+1)^{n+2} \subset \langle H_0, x\frac{\partial H_0}{\partial x}, \frac{\partial H_0}{\partial y} \rangle_{\mathcal{E}(r; n+1)} + \mathfrak{M}(1)\mathcal{E}(r; n+1). \tag{5.9}$$

Let $C' \in C_T^{\Theta, Z}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ satisfying $j^{n+4} C(0) = j^{n+4} C'(0)$ be given. There exists a function germ $H'(T, Q, p) \in \mathfrak{M}(2n+1)$ such that

$$P_{C'} = \{(T, Q, Z, -\frac{\partial H'}{\partial T}(T, Q, p) + s, -\frac{\partial H'}{\partial Q}, T, -\frac{\partial H'}{\partial p}, H' - \langle \frac{\partial H'}{\partial p}, p \rangle + Z, s, p)\}.$$

Since $H = z - qp$ on P_C and $H' = z - qp$ on $P_{C'}$, we have that $j^{n+4} H_0(0) = j^{n+4} H_0'(0)$, where $H_0'(x, y, t) = H'(t, x, 0, y) \in \mathfrak{M}(r; n+1)^2$. By (5.9) we have that

$$\mathfrak{M}(r; n)^{n+2} \subset \langle H_0, x\frac{\partial H_0}{\partial x}(x, y, 0), \frac{\partial H_0}{\partial y}(x, y, 0) \rangle_{\mathcal{E}(r; n)}$$

and this means that $H_0(x, y, 0)$ is reticular $\mathcal{K}$-$(n+3)$-determined by Lemma 2.1. Therefore we may assume that $H_0|_{t=0} = H_0'|_{t=0}$. It follows that $H_0 - H_0' \in \mathfrak{M}(1)\mathfrak{M}(r; n+1)^{n+3}$. Then

the function germ $G(x,y,t,q,z) = -z + H_0'(x,y,t) + \langle y,q \rangle \in \mathfrak{M}(r;n+1+n+1)$ is a generating family of $C'|_{\mathbb{L}}$.

We define the function germ $E_{\tau_0}(x,y,t,\tau) \in \mathcal{E}(r;n+1+1)$ by $E_{\tau_0}(x,y,t,\tau) = (1-\tau-\tau_0)H_0(x,y,t) + (\tau+\tau_0)H_0'(x,y,t)$ for $\tau_0 \in [0,1]$. By (5.8) and (5.9), we have that

$$\mathfrak{M}(r;n+1)^{n+3} \subset \langle H_0, x\frac{\partial H_0}{\partial x} \rangle_{\mathcal{E}(r;n+1)} + \mathfrak{M}(r;n+1)\langle \frac{\partial H_0}{\partial y} \rangle + \mathfrak{M}(1)\langle 1, y, \frac{\partial H_0}{\partial t} \rangle. \qquad (5.10)$$

Then we have that

$$\begin{aligned}
&\mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3} \mathcal{E}_{x,y,t,\tau} \\
&= \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3}(\mathcal{E}_{x,y,t} + \mathfrak{M}_\tau \mathcal{E}_{x,y,t,\tau}) \\
&\subset \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3} + \mathfrak{M}_t \mathfrak{M}_\tau \mathfrak{M}_{x,y,t}^{n+3} \mathcal{E}_{x,y,t,\tau} \\
&\subset \mathfrak{M}_t(\langle H_0, x\frac{\partial H_0}{\partial x} \rangle_{\mathcal{E}_{x,y,t}} + \mathfrak{M}_{x,y,t}\langle \frac{\partial H_0}{\partial y} \rangle + \mathfrak{M}_t \langle 1, y, \frac{\partial H_0}{\partial t} \rangle) + \mathfrak{M}_{t,\tau} \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3} \mathcal{E}_{x,y,t,\tau} \\
&\subset \mathfrak{M}_{t,\tau}\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x} \rangle_{\mathcal{E}_{x,y,t,\tau}} + \mathfrak{M}_{t,\tau} \mathfrak{M}_{x,y,t,\tau}\langle \frac{\partial E_{\tau_0}}{\partial y} \rangle \\
&\qquad\qquad + \mathfrak{M}_{t,\tau}^2 \langle 1, y, \frac{\partial E_{\tau_0}}{\partial t} \rangle + \mathfrak{M}_{t,\tau} \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3} \mathcal{E}_{x,y,t,\tau}.
\end{aligned}$$

By (5.10) we have the second inclusion. For the last inclusion, observe that

$$x_j \frac{\partial E_{\tau_0}}{\partial x_j} - x_j \frac{\partial H_0}{\partial x_j} = (\tau_0 + \tau)x_j \frac{\partial}{\partial x_j}(H_0' - H_0) \in \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3},$$

$$\frac{\partial E_{\tau_0}}{\partial y_j} - \frac{\partial H_0}{\partial y_j} = (\tau_0 + \tau)\frac{\partial}{\partial y_j}(H_0' - H_0) \in \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+2},$$

$$\frac{\partial E_{\tau_0}}{\partial t} - \frac{\partial H_0}{\partial t} = (\tau_0 + \tau)\frac{\partial}{\partial t}(H_0' - H_0) \in \mathfrak{M}_{x,y,t}^{n+3}.$$

By Malgrange preparation theorem we have that

$$\begin{aligned}
\frac{\partial E_{\tau_0}}{\partial \tau} &\in \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3} \subset \mathfrak{M}_t \mathfrak{M}_{x,y,t}^{n+3} \mathcal{E}_{x,y,t,\tau} \\
&\subset \mathfrak{M}_{t,\tau}(\langle E_{\tau_0}, x\frac{\partial E_{\tau_0}}{\partial x} \rangle_{\mathcal{E}_{x,y,t,\tau}} + \mathfrak{M}_{x,y,t,\tau}\langle \frac{\partial E_{\tau_0}}{\partial y} \rangle) + \mathfrak{M}_{t,\tau}^2 \langle 1, y, \frac{\partial E_{\tau_0}}{\partial t} \rangle.
\end{aligned}$$

for $\tau_0 \in [0,1]$. This means that there exist $\Phi(x,y,t) \in \mathcal{B}_1(r;n+1)$ and a unit $a \in \mathcal{E}(r;n+1)$ and $b_1(t), \ldots, b_n(t), c(t) \in \mathfrak{M}(1)$ such that
(1) Φ has the form: $\Phi(x,y,t) = (x\phi_1(x,y,t), \phi_2(x,y,t), \phi_3(t))$,
(2) $H_0(x,y,t) = a(x,y,t) \cdot H_0' \circ \Phi(x,y,t) + \sum_{i=1}^n y_i b_i(t) + c(t)$ for $(x,y,t) \in (\mathbb{H}^r \times \mathbb{R}^{n+1}, 0)$
    We define the reticular t-$\mathcal{P}$-$\mathcal{K}$-isomorphism (Ψ, d) by

$$\Psi(x,y,t,q,z) = (x\phi_1(x,y,t), \phi_2(x,y,t), \phi_3(t), q(1-b(t)), z), d(x,y,t,q,z) = a(x,y,t).$$

We set $G' := d \cdot G \circ \Psi \in \mathfrak{M}(r;n+n+1)$. Since $\frac{\partial E_{\tau_0}}{\partial \tau}|_{t=0} = 0$, we have that $a(x,y,0) = 1$ and $\Phi(x,y,0) = (x,y,0)$. Therefore we have that $G'(x,y,0,q,z) = -z + H_0(x,y,0) + \langle y,q \rangle = F(x,y,0,q,z)$ for $(x,y,q,z) \in (\mathbb{H}^r \times \mathbb{R}^{n+n+1}, 0)$. Then F and G' are reticular t-$\mathcal{P}$-$\mathcal{K}$-infinitesimal versal unfoldings of $F|_{t=0}$. Since G and G' are reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent, it

follows that F and G are reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent. Therefore $\mathcal{L}$ and $C'|_{\mathbb{L}}$ are $\mathcal{P}$-Legendrian equivalent. $\qquad\square$

Let $J^l(2n+3,2n+3)$ be the set of l-jets of map germs from $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ to $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ and $pC^l(n)$ be the Lie group in $J^l(2n+3,2n+3)$ consists of l-jets of $\mathcal{P}$-contact diffeomorphism germs on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$. Let $L^l(2n+3)$ be the Lie group consists of l-jet of diffeomorphism germs on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$.

We consider the Lie subgroup $rpLe^l(n)$ of $L^l(2n+3)\times L^l(2n+3)$ consists of l-jets of reticular $\mathcal{P}$-diffeomorphisms on the source space and l-jets of $\mathcal{P}$-Legendrian equivalences of Π:

$$rpLe^l(n)=\{(j^l\phi(0),j^lK(0))\in L^l(2n+3)\times L^l(2n+3)\mid \phi \text{ is a reticular}$$
$$\mathcal{P}\text{-diffeomorphism on } (J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0), K \text{ is a } \mathcal{P}\text{-Legendrian equivalence of } \Pi\}.$$

The group $rpLe^l(n)$ acts on $J^l(2n+3,2n+3)$ and $pC^l(n)$ is invariant under this action. Let C be a $\mathcal{P}$-contact diffeomorphism germ on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ and set $z=j^lC(0)$, $\mathcal{L}=C|_{\mathbb{L}}$. We denote the orbit $rpLe^l(n)\cdot z$ by $[z]$. Then

$$[z]=\{j^lC'(0)\in pC^l(n)\mid \mathcal{L} \text{ and } C'|_{\mathbb{L}} \text{ are } \mathcal{P}\text{-Legendrian equivalent}\}.$$

We denote by VI_C the vector space consists of infinitesimal $\mathcal{P}$-contact transformation germs of C and denote by VI_C^0 the subspace of VI_C consists of germs which vanish on 0. We denote by $VL_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}$ by the vector space consists of infinitesimal $\mathcal{P}$-Legendrian equivalences on Π and denote by $VL_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}^0$ by the subspace of $VL_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}$ consists of germs which vanish at 0. We denote by $V_{\mathbb{L}}^0$ the vector space consists of infinitesimal reticular $\mathcal{P}$-diffeomorphisms on $(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$ which vanishes at 0. By Lemma 4.3, we have that:

$$VI_C^0=\{v:(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\to(T(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})),0)\mid$$
$$v=X_f\circ C \text{ for some } f\in\mathfrak{M}_{t,q,z,p}^2\},$$
$$VL_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}^0=\{\eta\in X(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\mid$$
$$\eta=X_H \text{ for some } \mathcal{P}\text{-fiver preserving function germ } H\in\mathfrak{M}_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}^2\},$$
$$V_{\mathbb{L}}^0=\{\xi\in X(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\mid\xi=X_g \text{ for some } g\in B'\},$$

where $B'=\langle q_1p_1,\ldots,q_rp_r\rangle_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}+\mathfrak{M}_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}\langle q_{r+1},\ldots,q_n,z\rangle+\mathfrak{M}_t\langle s\rangle$.

We define the homomorphisms $tC:VI_{\mathbb{L}}^0\to VI_C^0$ by $tC(v)=C_*v$ and $wC:VL_{J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})}^0\to VI_C^0$ by $wC(\eta)=\eta\circ C$.

We denote VI_C^l the subspace of VI_C consists of infinitesimal $\mathcal{P}$-contact transformation germs of C whose l-jets are 0:

$$VI_C^l=\{v\in VI_C\mid j^lv(0)=0\}.$$

For $\tilde{C}\in C_T^\Theta(U,J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}))$, we define the continuous map $j_0^l\tilde{C}:U\to pC^l(n)$ by (T^0,Q^0,Z^0,S^0,P^0) to the l-jet of the map $(T,Q,Z,S,P)\mapsto\tilde{C}(T+T^0,Q+Q^0,Z+Z^0,S+S^0,P+P^0)+S^0T+P^0Q-\tilde{C}(T^0,Q^0,Z^0,S^0,P^0)$ at 0.

We also define $j_0^lC:(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)\to pC^l(n)$ by the same method for $C\in C_T^\Theta(J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),0)$.

Proposition 5.4 *Let $C \in C_T^{\Theta}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ and set $z = j^l C(0)$. Then $j_0^l C$ is transversal to $[z]$ if and only if*

$$tC(V_{\mathbb{L}}^0) + wC(VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})})) + VI_C^{l+1} = VI_C.$$

Proof. We consider the surjective projection $\pi_l : VI_C \to T_z(pC^l(n))$. Since $(j^l C)_*(v) = \pi_l(C_* v)$, it follows that $j^l C$ is transversal to $[z]$ if and only if $(j^l C)_*(T_0(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))) + T_z[z] = T_z(pC^l(n))$ and this holds if and only if

$$(\pi_l)^{-1}((j^l C)_*(T_0(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) + tC(V_{\mathbb{L}}^0) + wC(VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}^0) + VI_C^{l+1} = VI_C$$

and this holds if and only if

$$tC(V_{\mathbb{L}}^0) + wC(VL_{J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})}) + VI_C^{l+1} = VI_C.$$

$\square$

Theorem 5.5 *Let $\mathcal{L}$ be a reticular Legendrian unfolding. Let $C \in C_T^{\Theta}(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be an extension of $\mathcal{L}$ and $l \geq (n+2)^2$. Then the followings are equivalent:*
(s) *$\mathcal{L}$ is stable.*
(t) *$j_0^l C$ is transversal to $[j_0^l C(0)]$.*
(a') *$\mathcal{E}_{t,q,p} = B_0 + \langle 1, p_1 \circ C', \ldots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} + \langle s \circ C' \rangle_{\mathcal{E}_t} + \mathfrak{M}_{t,q,p}^l$, where $C' = C|_{z=s=0}$ and $B_0 = \langle q_1 p_1, \ldots, q_r p_r, q_{r+1}, \ldots, q_n \rangle_{\mathcal{E}_{t,q,p}}$,*
(a) *$\mathcal{E}_{t,q,p} = B_0 + \langle 1, p_1 \circ C', \ldots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} + \langle s \circ C' \rangle_{\mathcal{E}_t}$,*
(is) *$\mathcal{L}$ is infinitesimally stable.*

Proof. (s)$\Rightarrow$(t): Let $\tilde{C} \in C_T^{\Theta}(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ be a representative of C. By theorem 4.4 and (s), there exists $\tilde{C}' \in C_T^{\Theta}(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ such that $j_0^l \tilde{C}'$ is transversal to $[j_0^l C(0)]$, and $\tilde{C}'|_{\mathbb{L}}$ at $w = (t, 0, \ldots, 0, p_{r+1}, \ldots, p_n) \in U$ and $\mathcal{L}$ are $\mathcal{P}$-Legendrian equivalent. This means that $[j_0^l \tilde{C}_w'(0)] = [j^l C(0)]$ and hence $j_0^l C$ is transversal to $[j_0^l C(0)]$ at 0.
(t)$\Leftrightarrow$(a): This is proved by an analogous method of Theorem 4.5.
(a)$\Leftrightarrow$(a'): We need only to prove (a')$\Rightarrow$(a). By the restriction of (a') to $t = 0$ we have that:

$$\mathcal{E}_{q,p} = B_0' + \langle 1, p_1 \circ C'', \ldots, p_n \circ C'' \rangle_{(\Pi \circ C'')^* \mathcal{E}_{t,q,z}} + \langle s \circ C'' \rangle_{\mathbb{R}} + \mathfrak{M}_{q,p}^l,$$

where $C'' = C'|_{t=0}$ and $B_0' = B_0|_{t=0}$. Then we have that

$$\mathcal{E}_{q,p} = B_0' + (\Pi \circ C'')^* \mathfrak{M}_{t,q,z} \mathcal{E}_{q,p} + \langle 1, p_1 \circ C'', \ldots, p_n \circ C'', s \circ C'' \rangle_{\mathbb{R}} + \mathfrak{M}_{q,p}^l.$$

It follows that

$$\mathfrak{M}_{q,p}^{n+2} \subset B_0' + (\Pi \circ C'')^* \mathfrak{M}_{t,q,z} \mathcal{E}_{q,p}.$$

Therefore

$$\mathfrak{M}_{t,q,p}^{n+2} \subset B_0 + (\Pi \circ C')^* \mathfrak{M}_{t,q,z} \mathcal{E}_{q,p} + \mathfrak{M}_t \mathcal{E}_{t,q,p},$$

and we have that

$$\mathfrak{M}_{t,q,p}^l = (\mathfrak{M}_{t,q,p}^{n+2})^{n+2} \subset B_0 + (\Pi \circ C')^* \mathfrak{M}_{t,q,z}^{n+2} \mathcal{E}_{t,q,p} + \mathfrak{M}_t \mathcal{E}_{t,q,p}.$$

It follows that

$$\mathcal{E}_{t,q,p} = B_0 + \langle 1, p_1 \circ C', \ldots, p_n \circ C' \rangle_{(\Pi \circ C')^* \mathcal{E}_{t,q,z}} + \langle s \circ C' \rangle_{\mathcal{E}_t} +$$
$$(\Pi \circ C')^* \mathfrak{M}_{t,q,z}^{n+2} \mathcal{E}_{t,q,p} + \mathfrak{M}_t \mathcal{E}_{t,q,p}.$$

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This means (a) by [12, Corollary 1.8].

(a)$\Leftrightarrow$(is): This is proved in Theorem 4.5.

(t)&(is)$\Rightarrow$(s): Since $j_0^l C$ is transversal to $[j_0^l C(0)]$, it follows that there exists a representative $\tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ of C and a neighbourhood $W_{\tilde{C}}$ of $\tilde{C}$ in $C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ such that for any $\tilde{C}' \in W_{\tilde{C}}$ there exists $w \in U$ such that $j_0^l \tilde{C}'$ is transversal to $[j_0^l C(0)]$ at w. Since $j_0^l \tilde{C}'_w(0) \in [j_0^l C(0)]$, it follows that there exists $C'' \in C_T^\Theta(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that $\mathcal{L}$ and $C''|_{\mathbb{L}}$ are $\mathcal{P}$-Legendrian equivalent and $j_0^l C''(0) = j_0^l \tilde{C}'_w(0)$. Since $\mathcal{L}$ is infinitesimally stable, it follows that $\mathcal{L}$ is $(n+5)$-determined by Theorem 5.3. Therefore we have that $C''|_{\mathbb{L}}$ is also $(n+5)$-determined. Then $C''|_{\mathbb{L}}$ and $\tilde{C}'_w|_{\mathbb{L}}$ are $\mathcal{P}$-Legendrian equivalent. This means that $\mathcal{L}$ is stable. $\qquad\square$

Let $\mathcal{L}$ be a stable reticular Legendrian unfolding. We say that $\mathcal{L}$ is *simple* if there exists a representative $\tilde{C} \in C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ of a extension of $\mathcal{L}$ such that $\{\tilde{C}_w | w \in U\}$ is covered by finite orbits $[C_1], \ldots, [C_m]$ for some $C_1, \ldots, C_m \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$.

Lemma 5.6 *Let $\mathcal{L}$ be a stable reticular Legendrian unfolding and $l \geq (n+2)^2$. Let $C \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ be an extension of $\mathcal{L}$. Then $\mathcal{L}$ is simple if and only if there exists an open neighborhood W_z of $z = j_0^l C(0)$ in $pC^l(n)$ and $z_1, \ldots, z_m \in pC^l(n)$ such that $W_z \subset [z_1] \cup \cdots \cup [z_m]$.*

Proof. Suppose that $\mathcal{L}$ is simple. Then there exists a representative $\tilde{C} \in C_T(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ of a extension of $\mathcal{L}$ and $C_1, \ldots, C_m \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that

$$\{\tilde{C}_w | w \in U\} \subset [C_1] \cup \cdots \cup [C_m]. \tag{5.11}$$

Since $\mathcal{L}$ is stable, it follows that $j_0^l \tilde{C}$ is transversal to $[z]$ at 0 by Theorem 5.5. This means that there exists a neighbourhood W_z of z in $pC^l(n)$ such that $W_z \subset \cup_{w \in U} [j_0^l \tilde{C}(w)]$. It follows that $W_z \subset [j^l C_1(0)] \cup \cdots \cup [j^l C_m(0)]$.

Conversely suppose that there exist a neighbourhood W_z of z in $pC^l(n)$ and $z_1, \ldots, z_m \in pC^l(n)$ such that $W_z \subset [z_1] \cup \cdots \cup [z_m]$. Since the map $j_0^l \tilde{C} : U \to pC^l(n)$ is continuous, there exists a neighbourhood U' of 0 in U such that $j_0^l \tilde{C}(w) \in W_z$ for any $w \in U'$. Then we have that $\cup_{w \in U'} j_0^l \tilde{C}(x) \subset [z_1] \cup \cdots \cup [z_m]$. Choose $\mathcal{P}$-contact diffeomorphism germs $C_1, \ldots, C_m$ on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ such that $j^l C_j(0) = z_i$ for $i = 1, \ldots, m$. By Theorem 5.5 (a'), we may assume that each reticular Legendrian unfolding $C_i|_{\mathbb{L}}$ is stable, thus l-determined. For any $w \in U'$ there exists $i \in \{1, \ldots, m\}$ such that $j_0^l \tilde{C}(w) \in [j^l C_i(0)]$. It follows that reticular Legendrian unfoldings $\tilde{C}_w|_{\mathbb{L}}$ and $C_i|_{\mathbb{L}}$ are $\mathcal{P}$-Legendrian equivalent. Therefore $\tilde{C}_w \in [C_i]$. We have (5.11). $\qquad\square$

Lemma 5.7 *A stable reticular Legendrian unfolding $\mathcal{L}$ is simple if and only if for a generating family $F(x, y, t, q, z) \in \mathfrak{M}(r; k+1+n+1)$ of $\mathcal{L}$, $f(x, y) = F(x, y, 0, 0) \in \mathfrak{M}(r; k)^2$ is a $\mathcal{K}$-simple singularity.*

Proof. Suppose that $\mathcal{L}$ is simple. Then $\Pi \circ \mathcal{L}$ is simple as a reticular Legendrian map. It follows that f is a $\mathcal{K}$-simple singularity by [9, Lemma 13.7].

Conversely suppose that f is $\mathcal{K}$-simple. Let $l \geq (n+2)^2$. There exist a neighbourhood W_z of $z = j^l f(0)$ in $J^l(r+k, 1)$ and $f_1, \ldots, f_m \in \mathcal{E}(r; k)$ such that $W_z \subset [j^l f_1(0)] \cup \cdots \cup [j^l f_m(0)]$. We may assume that each f_i are simple and reticular $\mathcal{K}$-l-determined. We choose a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding $F_i^1(x, y, t, q, z) \in \mathfrak{M}(r; k+1+n+1)$ of f_i for each i. If there exists a

reticular $\mathcal{P}$-$\mathcal{K}$-stable unfolding as of f_i as $(n+2)$-dimensional unfolding, we set it by F_i^0, otherwise we set $F_i^0 = F_i^1$ for each i. We also choose an extension $C_i^j \in C_T(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ of a reticular Legendrian unfolding of which F_i^j is a generating family for each i, j. Let C_0 be an extension of $\mathcal{L}$. We may assume that the canonical relation P_{C_0} has the form:

$$P_{C_0} = \{(T, Q, Z, -\frac{\partial H_{C_0}}{\partial T}(T, Q, p) + s, -\frac{\partial H_{C_0}}{\partial Q}, T, -\frac{\partial H_{C_0}}{\partial p}, H_{C_0} - \langle \frac{\partial H_{C_0}}{\partial p}, p \rangle + Z, s, p)\}$$

for a function germ H_{C_0}. Then the l-jet of H_{C_0} is determined by the l-jet of C_0 since $H_{C_0} = z - qp$ on P_{C_0}. For a $\mathcal{P}$-contact diffeomorphism germ C on $(J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ around C_0, there exists a function germ $H_C(T, Q, p)$ satisfying the above condition for P_C. We define $H'_C(x, y) \in \mathfrak{M}(r; n)$ by $H'_C(x, y) = H_C(0, x, 0, y)$. Then there exists a neighbourhood U of $j^l C_0(0)$ such that the following maps are constructed:

$$\begin{array}{ccccc} U & \to & J^l(1+n+n, 1) & \to & J^l(r+n, 1) \\ j^l C(0) & \mapsto & j^l H_C(0) & \mapsto & j^l H'_C(0) \end{array}.$$

Then H'_{C_0} is reticular $\mathcal{K}$-equivalent to f. It follows that $j^l H'_{C_0}(0) \in W_z$. We set U' the inverse image of W_z by the above maps. For any $j^l C(0) \in U'$, there exists a number i such that $j^l H'_C(0) \in [f_i]$. By Theorem 5.5 (a'), we may assume that $C|_{\mathbb{L}}$ is a stable reticular Legendrian map. Since f_i is reticular $\mathcal{K}$-l-determined, we have that H'_C and f_i are reticular $\mathcal{K}$-equivalent. Then the reticular Legendrian map $C|_{\mathbb{L}}$ is $\mathcal{P}$-Legendrian equivalent to $C_i^0|_{\mathbb{L}}$ or $C_i^1|_{\mathbb{L}}$ and it follows that $j^l C(0) \in [C_i^0] \cup [C_i^1]$. Then we have that

$$U_z \subset [C_1^0] \cup [C_1^1] \cup \cdots \cup [C_m^0] \cup [C_m^1]$$

and this means that $\mathcal{L}$ is simple. $\qquad \square$

By [10, Proposition 6.5], we have that:

Theorem 5.8 *Let* $F(x, y, t, q, z) \in \mathfrak{M}(r; k+n+1)$ *be a* $\mathcal{P}$-C-*non-degenerate function germ for* $r = 0, n \leq 4$ *or* $r = 1, n \leq 2$. *Then* F *is stably reticular* t-$\mathcal{P}$-$\mathcal{K}$-*equivalent for one of the following types.*

In the case $r = 0, n \leq 4$: (^0A_l) $y_1^{l+1} + \sum_{i=1}^{l} q_i y_1^i + z$ $(2 \leq l \leq n)$,

$(^0D_4^\pm)$ $y_1^2 y_2 \pm y_2^3 + q_1 y_2^2 + q_2 y_2 + q_3 y_1 + z$,
$(^0D_5)$ $y_1^2 y_2 + y_2^4 + q_1 y_2^3 + q_2 y_2^2 + q_3 y_2 + q_4 y_1 + z$,

(^1A_l) $y_1^{l+1} + (t + q_l^2 \pm q_{l+1}^2 \pm \cdots \pm q_n^2) y_1^{l-1} + \sum_{i=1}^{l-1} q_i y_1^i + z$ $(3 \leq l \leq n)$,

$(^1D_4^\pm)$ $y_1^2 y_2 \pm y_2^3 + t y_2^2 + q_1 y_2 + q_2 y_1 + z$, $y_1^2 y_2 \pm y_2^3 + (t + q_3^2) y_2^2 + q_1 y_2 + q_2 y_1 + z$,
$(^1D_5)$ $y_1^2 y_2 + y_2^4 + t y_2^3 + q_1 y_2^2 + q_2 y_2 + q_3 y_1 + z$, $y_1^2 y_2 + y_2^4 + (t + q_4^2) y_2^3 + q_1 y_2^2 + q_2 y_2 + q_3 y_1 + z$,
$(^1D_6^\pm)$ $y_1^2 y_2 \pm y_2^5 + t y_2^6 + q_1 y_2^3 + q_2 y_2^2 + q_3 y_2 + q_4 y_1 + z$,
$(^1E_6)$ $y_1^3 + y_2^4 + t y_1 y_2^2 + q_1 y_1 y_2 + q_2 y_2^2 + q_3 y_1 + q_4 y_2 + z$.
In the case $r = 1, n \leq 2$: $(^0B_2)$ $x^2 + q_1 x + z$,
$(^0B_3)$ $x^3 + q_1 x^2 + q_2 x + z$,
$(^0C_3^\pm)$ $\pm xy + y^3 + q_1 y^2 + q_2 y + z$,
$(^1B_3)$ $x^3 + t x^2 + q_1 x + z$, $x^3 + (t + q_2^2) x^2 + q_1 x + z$,
$(^1B_4)$ $x^4 + t x^3 + q_1 x^2 + q_2 x + z$,

$(^1C_3^\pm)$ $\pm xy + y^3 + ty^2 + q_1 y + z$, $\pm xy + y^3 + (t + q_2^2)y^2 + q_1 y + z$,
$(^1C_4)$ $xy + y^4 + ty^3 + q_1 y^2 + q_2 y + z$,
$(^1F_4)$ $x^2 + y^3 + txy + q_1 x + q_2 y + z$.

Theorem 5.9 *Let $r = 0, n \le 4$ or $r = 1, n \le 2$. Let U be a neighbourhood of 0 in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then there exists a residual set $O \subset C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}))$ such that for any $\tilde{C} \in O$ and $w \in U$, the reticular Legendrian unfolding $\tilde{C}_w|_\mathbb{L}$ is stable and have a generating family which is stably reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalent for one of the types in the previous theorem.*

Proof. In the case $r = 1$, $n \le 2$. Let $F_X(x, y, t, q) \in \mathfrak{M}(r; k+1+n)$ be a reticular t-$\mathcal{P}$-$\mathcal{K}$-stable unfolding of singularity $X \in \mathfrak{M}(r; k)^2$ for

$$X = B_2, B_3, B_4, C_3^\pm, C_4, F_4.$$

Then other unfoldings are not stable since other singularities have reticular $\mathcal{K}$-codimension > 4. We choose stable reticular Legendrian unfoldings $\mathcal{L}_X : (\mathbb{L}, 0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), 0)$ with the generating family F_X, and C_X be an extension of $\mathcal{L}_X$ for above list. Let $l > 16$. We define that

$$O' = \{\tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \mid j_0^l \tilde{C} \text{ is transversal to } [j^l C_X(0)] \text{ for all } X\}.$$

Then O' is a residual set. We set

$$Y = \{j^l C(0) \in C^l(n) \mid \text{the codimension of } [j^l C(0)] > 2n + 4\}.$$

Then Y is an algebraic set in $pC^l(n)$. Therefore we can define that

$$O'' = \{\tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \mid j_0^l \tilde{C} \text{ is transversal to } Y\}.$$

Then Y has codimension $> 2n + 4$ because all $\mathcal{P}$-contact diffeomorphism germ with $j^l C(0) \in Y$ adjoin to the above list which are simple. Therefore

$$O'' = \{\tilde{C} \in C_T^\Theta(U, J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})) \mid j_0^l \tilde{C}(U) \cap Y = \emptyset\}.$$

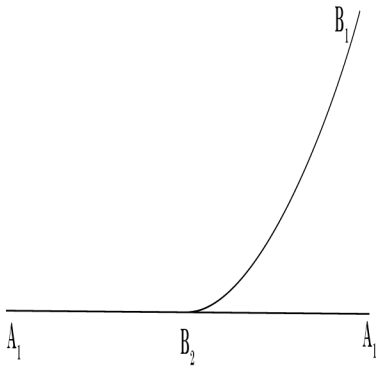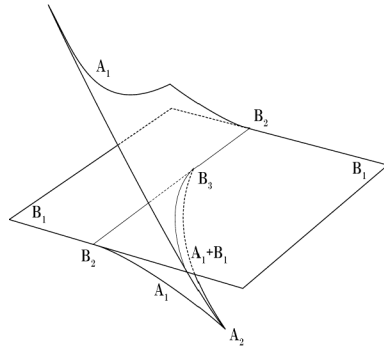Then the set $O = O' \cap O''$ has the required condition. $\square$
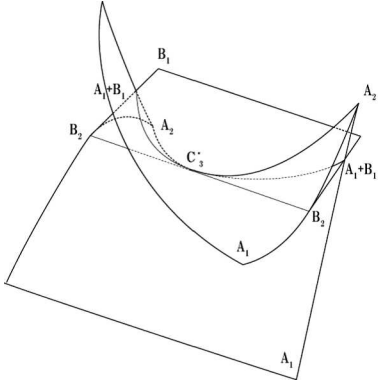

Figure 3: $^0B_2$
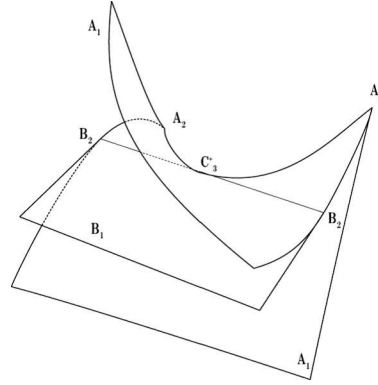

Figure 4: $^0B_3$
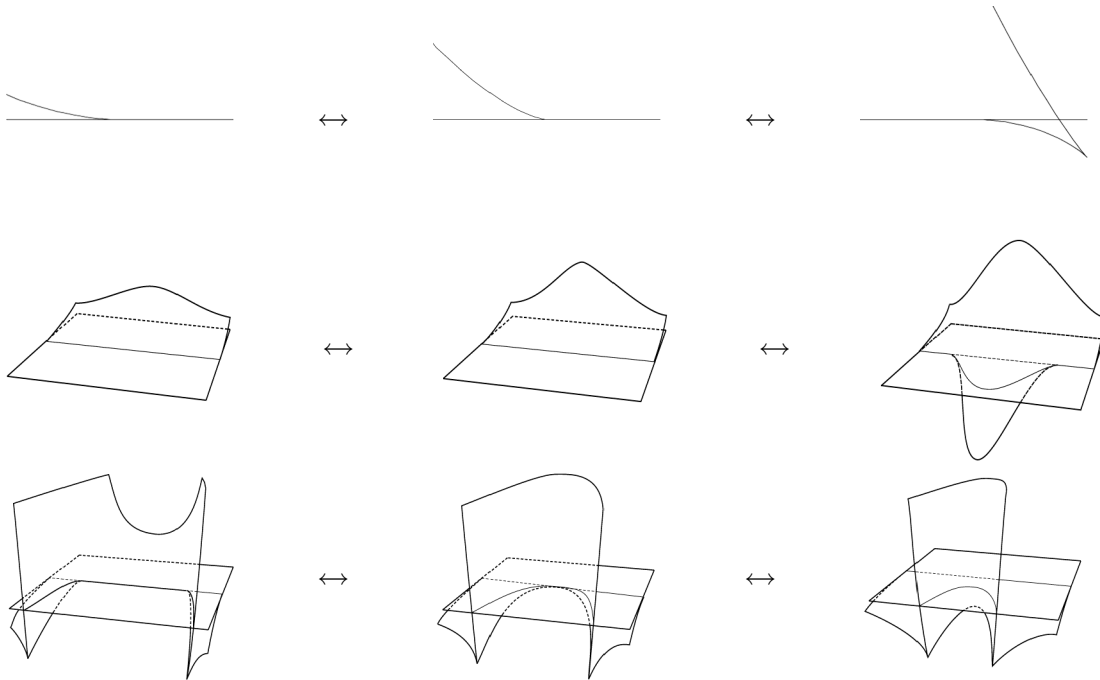
23

Figure 5: $^0C_3^-$

Figure 6: $^0C_3^+$
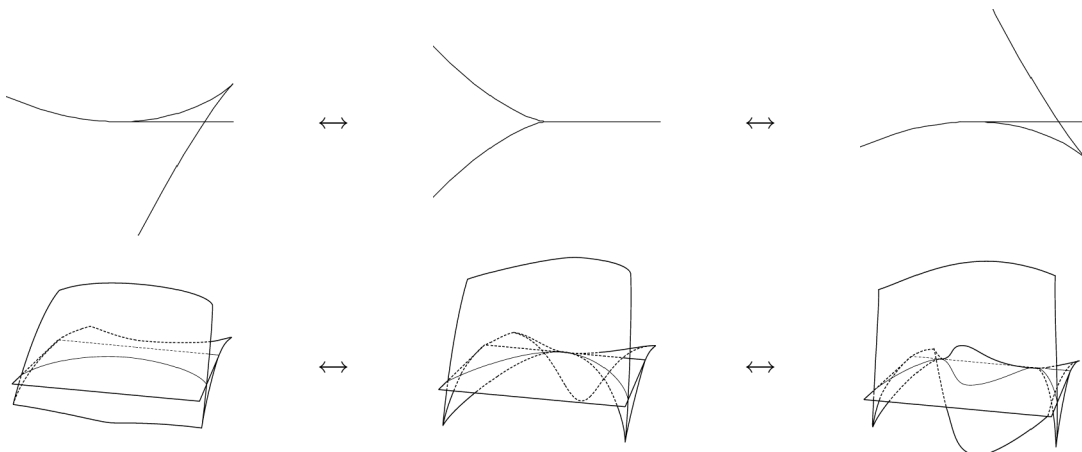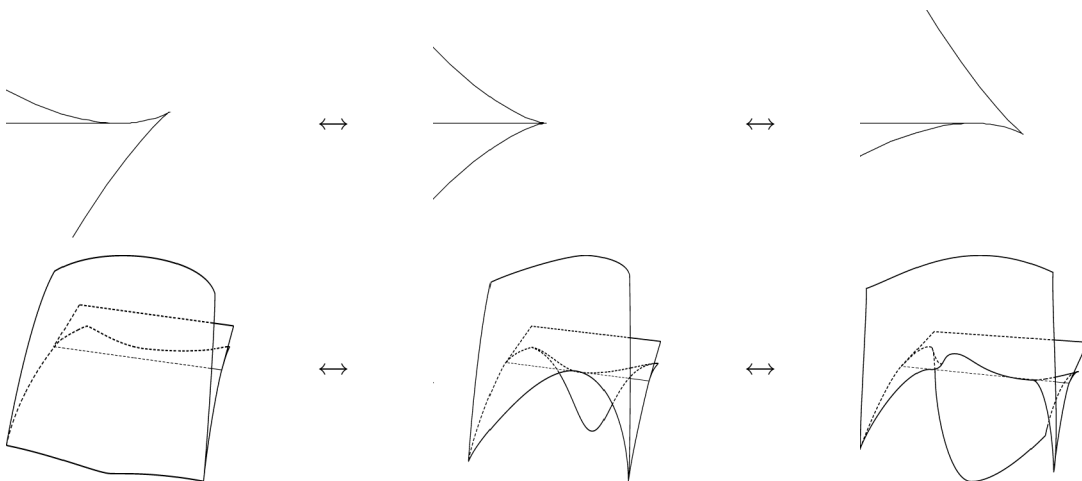

Figure 7: $^1B_3$

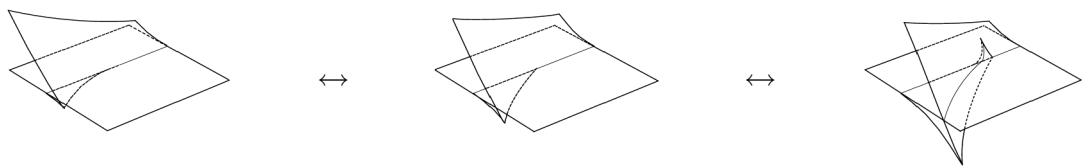24

Figure 8: $^1C_3^-$

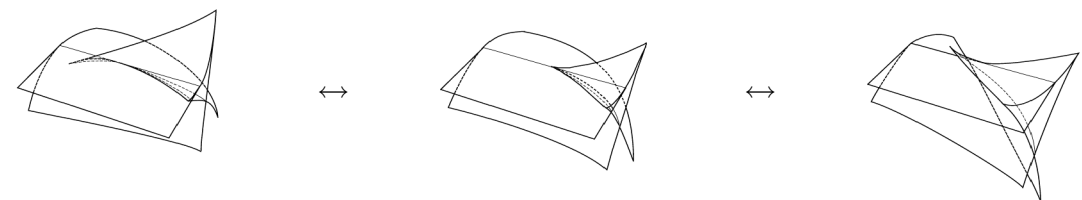Figure 9: $^1C_3^+$
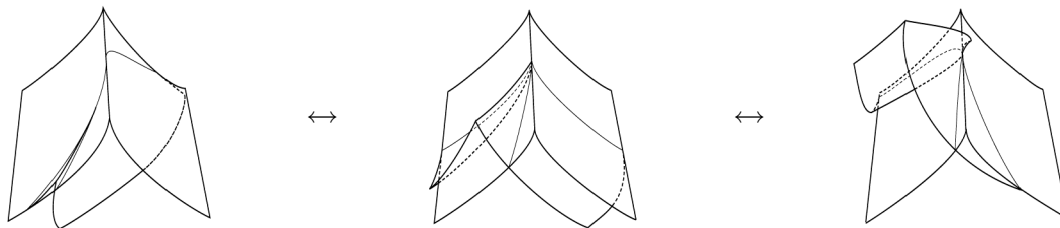

Figure 10: $^1B_4$

Figure 11: $^1C_4$

Figure 12: $^1F_4$

References

[1] V.I.Arnold,S.M. Gusein-Zade & A.N. Varchenko, *Singularities of differential maps,* Vol.I, Birkh user, Basel,1986.

[2] S.Izumiya and G.T.Kossioris, *Semi-local Classification of Geometric Singularities for Hamilton-Jacobi Equations*, J. of Differential Equations., 118, 166-193, 1995.

[3] S.Izumiya, *Perestroikas of Optical Wave fronts and Graphlike Legendrian Unfoldings* , J. Differential Geom., 38, 485-500, 1993.

[4] Th:bröcker, *Differentiable Germs and Catastrophes*, London Math. Soc. Lecture Note ser., 17 Cambridge Univ. Press, 1975.

[5] K.Jänich, *Caustics and catastrophes*, Math. Ann., 209, 161-180, 1974.

[6] T.Tsukada, *Reticular Lagrangian, legendrian Singularities and their applications*, PhD Thesis, Hokkaido University, 1999.

[7] T.Tsukada, *Reticular Lagrangian Singularities*, The Asian J. of Math., vol 1, 3, 572-622, 1997.

[8] T.Tsukada, *Reticular Legendrian Singularities*, The Asian J. of Math., 1, 109-127, 2001.

[9] T.Tsukada, *Genericity of Caustics and Wavefronts on an r-corner*, in preparation.

[10] T.Tsukada, *A generic classification of function germs with respect to the reticular t-$\mathcal{P}$-$\mathcal{K}$-equivalence*, Hokkaido Math. J., 38, 177-203, 2009.

[11] G.Wassermann, *Stability of unfolding*, Lecture note in mathematics, 393.

[12] G.Wassermann, *Stability of unfolding in space and time*, Acta Math., 135, 57-128, 1975.

[13] V.M.Zakalyukin, *Reconstruction of fronts and caustics depending on a parameter and versality of mappings*, J. of Soviet Math., 27, 2713-2735, 1984.